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SANTILLI'S ISOTOPIES of CONTEMPORARY ALGEBRAS, GEOMETRIES and RELATIVITIES

Second Edition

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SANTILLI'S ISOTOPIES OF CONTEMPORARY ALGEBRAS, GEOMETRIES AND RELATIVITIES

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НАУКОВА ДУМКА

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J. V. Kadeisvili

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PART II: LIE–SANTILLI ISOTHEORY (1996)

FOREWORD OF THE FIRST EDITION

This book originates from two recent articles by Prof. Ruggero Maria Santilli appeared in *Algebras, Groups and Geometries* (Santilli (1991a, b)) on his isotopic liftings of contemporary mathematical structures.

I thought that their rewriting in the form of a short monograph, with the addition of a few complementary aspects and applications, may be useful for applied mathematicians and theoretical physicists interested in examining Prof. Santilli's novel mathematical and physical theories.

I would like to thank Prof. Santilli for invaluable assistance and for authorizing free use of his computer disks in the preparation of this volume.

J. V. Kadeisvili
Spring 1992

FOREWORD OF THE SECOND EDITION

Part I of this second edition, on Santilli's isotopies of contemporary algebras, geometries and isorelativities, consists of the entire first edition with numerous corrections following comments I received from various readers, which are here gratefully appreciated.

Part II is new and deals with the nonlinear, nonlocal and nonhamiltonian isotopies of the various branches of Lie's theory today known as the *Lie-Santilli isotheory*. This Part II also includes an updated and more advanced formulation of the isotopies and isodualities of contemporary mathematics.

Part III has been added while this volume was about to be released for print, and outlines the *iso-grand-unification* and *isocosmology* to be presented by at the forthcoming *VIII M. Grossmann Meeting on General Relativity* scheduled in Jerusalem this coming June, 1997.

Part I is recommended for a first study of the isotopies of contemporary algebras, geometries and relativities. Part II is recommended as a more advanced study of the topics. The understanding of Part III requires an advanced technical knowledge of the field.

I left the references of Part I unchanged and added new references for Parts II and III. In this way, the reader can also see the rather remarkable progress made in the field during the past five years.

This volume is primarily devoted to theoretical profiles. For the numerous applications and experimental verifications of isotopic theories and their isoduals today available in particles physics, nuclear physics, astrophysics, superconductivity, biology and other fields, we suggest Ref. [101] of Part II.

Again, I have no words to express my thanks to Prof. Santilli for his guidance and support as well as for allowing me to use his computer disks. Additional thanks are due to Mrs. Pamela Fleming of the Institute for Basic Research for logistic assistance and to various colleagues for critical comments and suggestions.

J. V. Kadeisvili
Spring 1997

PART I:

SANTILLI'S ISORELATIVITIES

1992

I.1: STATEMENT OF THE PROBLEM

In a series of contributions made over the past two decades, ***Ruggero Maria Santilli***[1] has achieved certain isotopic gene-realizations of Galilei's, Einstein's special and Einstein's general relativities, today called *Santilli's isotopic relativities* [2], which:

I) have been conceived for the description of systems beyond the representation capabilities of conventional relativities, the nonlinear, nonlocal, nonlagrangian, nonhamiltonian and non-Newtonian systems[3] characterizing

[1] See Santilli's contributions listed in the bibliography from 1967 to 1991.

[2] See the monograph by Aringazin *et al.* (1991), or the review article by Aringazin *et al.* (1992), as well as a number of research papers quoted later on in the text.

[3] Santilli uses the following terminology which is also adopted in this monograph:

a) *Nonlinearity* is referred to all variables and quantities considered, as well as to all their possible or otherwise needed derivatives (this implies that, at the operator level, Santilli's notion of nonlinearity is referred, not only to the wavefunctions but also, and perhaps more importantly, to their derivatives);

b) *Nonlocality* is referred to interactions requiring an essential representation via surface or volume integrals;

c) *Lagrangians* are specifically intended to be *first-order Lagrangians* , i.e., functions L depending on time t, the local coordinates r and their derivatives of the first-order only, $L = L(t, r, \dot{r})$. A given system is then called *nonlagrangian* when it cannot be represented in terms of a first-order Lagrangian in the given r-frame, but it may admit representations in terms of higher-order Lagrangians $L(t, \dot{r}, \ddot{r}, ...)$.

d) *Hamiltonians* are intended the conventional functions $H(t, r, p)$ obtained via the usual Legendre transform of a first-order Lagrangian $L(t, r, \dot{r})$. A given system is then called *nonhamiltonian* when it cannot be represented via the conventional Hamiltonian $H(t, r, p)$ in the given r-frame, but it may admit representations via generalized formulations equivalent to higher-order Lagrangians (such as the Birkhoffian mechanics of Sect. 7).

e) *Newtonian forces* are generally referred to forces F depending on time, coordinates and velocities, $F = F(t, r, \dot{r})$. Santilli calls a force F *non-Newtonian* , when it also depends on derivatives higher than the first, $F = F(t, r, \dot{r}, \ddot{r}, ...)$.

The reader should therefore keep in mind that *Santilli's isotopic relativities characterize the most general possible systems which can be identified via current*

motion of extended particles within physical media;

II) are based on the isotopic generalization of the virtual entirety of contemporary mathematical structures, including: fields, vector spaces, transformations, Lie algebras, Hamiltonian mechanics, Lie symmetries, and the symplectic, affine and Riemannian geometries;

III) are a covering of the conventional relativities, because admitting the latter as a particular case when motion returns to be in vacuum (empty space), and all acting forces return to be local-differential and conservative.

By using a language accessible to both mathematicians and theoretical physicists, in this monograph, I shall review Santilli's generalized mathematical structures because they are mathematically relevant *per se* , and then outline their application for the construction of the isotopic relativities.

The conditions of exact applicability of conventional relativities are those of their original conception[4], namely, particles which can be well approximated as being point-like while moving in the homogeneous and isotropic vacuum, under action at-a-distance, potential and therefore, *variationally selfadjoint* (SA) forces (Helmholtz (1887), Santilli (1978e)).

These physical conditions were historically referred to by Lagrange (1788), Hamilton (1834), Jacobi (1837), and other Founders of contemporary analytic dynamics as those of the *exterior dynamical problem*, namely, the study of dynamics in the empty space outside the minimal surface containing all matter of the body considered, including its possible atmosphere. A typical example is given by a satellite in orbit around Earth.

Mathematically, the above conditions render exactly applicable the conventional local-differential structures of contemporary mathematics in their canonical-Hamiltonian realizations, such as algebras, geometries, analytic mechanics, etc.

As indicated before, Santilli's isotopic relativities have been worked out for physical conditions fundamentally more general than the above ones and, in

mathematical and physical knowledge, including discrete systems, as shown by Jannussis et al. (1982), (1985), (1986).

[4] For historical presentations one may consult Galilei (1638), Newton (1687), Lorentz (1904), Poincaré (1905), Einstein (e.g., (1905) and (1916)), Minkowski (1913), and others. For a comprehensive historical bibliography on the special and general relativities one may consult Pauli (1921), in the English edition of 1981. Contemporary formulations of Galilei's relativity can be found in Levy-Leblond (1971), or Sudashan and Mukunda (1974). Among a number of contemporary presentations of Einstein's relativities so large to discourage an outline, this author still prefers Pauli (*loc. cit.*) for reasons of completeness that will transpire in the geometrical parts of this volume (e.g., because Pauli indeed reviews the Freud (1939) identity of the Riemannian geometry, which is generally ignored by contemporary presentations, and other reasons).

particular, for conditions historically referred to as those of the *interior dynamical problem* , namely, the study of dynamics in the interior of the minimal surface encompassing all matter of the body considered. A typical example is given by a satellite during re-entry in Earth's atmosphere.

In fact, Lagrange (*loc. cit.*) and Hamilton (*loc. cit.*) formulated their original analytic equations *with external terms* precisely to represent the contact interior forces which were known since their time to be outside the representational capabilities of the Lagrangian or Hamiltonian. It was essentially at the end of the past century that, as a result of the work by Lie (1893) and for other historical reasons, the original Lagrange's and Hamilton's equations were "truncated" with the removal of the external terms, by acquiring the form of conventional use in contemporary mathematics and physics.

From a mathematical viewpoint, the latter systems can be represented as follows. Denote with $T^*E(r,\delta,\Re)$ the cotangent bundle of the three-dimensional Euclidean space $E(r,\delta,\Re)$ with local chart r and metric $\delta = \mathrm{diag.}\ (1,1,1)$ over the reals $\Re$. Then, the isotopic relativities provide a form-invariant description of systems of N particles of mass $m_a \neq 0$, $a = 1, 2, ..., N$, in their first-order, vector-field form which can be written

$$a = (a^{\mu}) = \begin{pmatrix} \dot{r}^{ia} \\ \dot{p}_{ia} \end{pmatrix} = (\Gamma^{\mu}(t, a, \dot{a}, ..)) = \Gamma =$$

$$= \begin{pmatrix} p_{ia}/m_a \\ F^{SA}_{ia}(r) + F^{NSA}_{ia}(t, r, p, \dot{p}, ...) + \int_{\sigma} d\sigma\, \mathfrak{F}^{NSA}_{ia}(t, r, p, \dot{p}, ...) \end{pmatrix}, \qquad (1.1)$$

$$i = 1, 2, 3\ (= x, y, z), \quad a = 1, 2, ..., N, \quad \mu = 1, 2, ... 6N.$$

where: the r's are the coordinates of the experimenter, the p's represent the linear momenta; the m's are the masses of the N particles; SA and NSA stand for variational selfadjointness and nonselfadjointness (Santilli (1978e)), respectively; and σ represents a surface or volume.

The objective of this monograph is to review the generalized mathematical tools used by Santilli for the treatment of systems (1.1). The following additional introductory comments appear to be recommendable.

Recall that Einstein did not claim Galilei's relativity to be *"violated"* for very high speed, but merely *"inapplicable"*. In this way, he constructed a covering of Galilei's relativity admitting of the latter at low speeds.

Along the same historical teaching, Santilli stresses that conventional relativities are not *"violated"* for the systems considered, but merely *inapplicable*,

because we are dealing with physical conditions substantially beyond those of their original conception. He then constructed isotopic coverings of the conventional relativities which admit the latter identically when their physical conditions are recovered.

The inapplicability of the conventional relativities for Santilli's broader conditions is beyond any meaningful doubt. As an example, Einstein's special and general relativities are inapplicable to systems (1.1) because of:

a) the inapplicability of their local-differential topology (e.g., the Zeeman topology of the special relativity), due to the nonlocal-integral character of the broader systems considered;

b) the inapplicability of their Lagrangian character, because the systems considered are nonselfadjoint by conception and experimental evidence;

c) the inapplicability of their canonical-Hamiltonian formulation, due to the nonhamiltonian character of the systems;

d) the inapplicability of their homogeneous and isotropic structure, owing to the physical evidence that the material media of interior problems, such as our atmosphere, are generally *inhomogeneous* (e.g., because of the local variation of the density) and *anisotropic* (e.g., because of the intrinsic angular momenta of Earth which evidently creates a preferred direction in the medium itself);

e) the inapplicability of Galilei's, Lorentz's and Poincaré's symmetries, due to numerous independent reasons, such as their strictly linear and local characters, as compared to the necessarily nonlinear and nonlocal character of systems (1.1);

f) the inapplicability of the conservation laws of physical quantities, because of, e.g., the monotonically decreasing character of the angular momentum of the space-ship during re-entry in Earth's atmosphere, which is contrary to the fundamental conservation laws of established relativities;

g) the inapplicability of the conventional symplectic geometry (see, e.g., Abraham and Marsden (1967)), affine geometry (see, e.g., Schrödinger (1950)) and Riemannian (1868) geometry (see, e.g., Lovelock and Rund (1975)), trivially, because of their local-differential character as compared to the nonlocal-integral nature of the systems considered;

and several other mathematical, theoretical and experimental reasons.

Santilli illustrates rather convincingly all the above occurrences by considering a limit case of interior problems, such as the core of a star undergoing gravitational collapse. In this case we have the mutual penetration of a very large number of wavepackets of the particle constituents and their compression in a very small region of space. The emergence of nonlinear, nonlocal and nonlagrangian-nonhamiltonian forces under these conditions, and the consequential inapplicability of the Riemannian geometry

are simply beyond any credible doubt. It is hoped the reader sees in this way the inapplicability of Einstein's gravitation for a more adequate study of interior gravitational problems, as manifest via a mere inspection of systems (1.1) (see Sect. 12 for more details). Santilli nevertheless stresses that the Riemannian geometry remains exactly valid for the physical conditions of its original use: the exterior gravitational problem of test particle moving in empty space.

This establishes the need to identify mathematical tools for the effective treatment of systems (1.1), and the consequential physical need to build a new generation of covering relativities. For more details on Santilli's isotopic relativities see the two recent monographs (Santilli (1991a, b)) and the review monograph (Aringazin et *al.* (1991)).

To begin our review, let us recall that Santilli's studies are based on the so-called *isotopies* of conventional formulations, which essentially are characterized by the *liftings from the conventional to the most general possible, nonlinear and nonlocal, axiom-preserving formulations of current mathematical structures*.

The fundamental mathematical (and physical) idea (Santilli (1978), (1979), (1980), and others) is the generalization of the conventional trivial, n-dimensional *unit* 1 of current theories, $1 = \mathrm{diag.}\ (1, 1, ..., 1)$, into a quantity $\hat{1}$ called *isotopic unit*, or *isounit*, which is nowhere null in the considered region of the local variables, and Hermitean, but otherwise possesses the most general possible, nonlinear and nonlocal dependence on: coordinates r; their derivatives of arbitrary order $\dot{r}, \ddot{r}, ...$ (or $p, \dot{p}, ...$); as well as any other needed quantity, such as the density $\mu = \mu(r)$ of the local medium considered, its local temperature $\tau(r)$, its index of refraction $n = n(r)$ (if any), etc.

$$\hat{1} = \hat{1}(t, r, \dot{r}, \ddot{r}, \mu, \tau, n, ...). \qquad (1.2)$$

All contemporary mathematical structures, such as fields, vector spaces, transformation theory, algebras, analytic mechanics, symplectic geometry, affine geometry, Riemannian geometry, etc. were then generalized by Santilli in such a way to admit the quantity $\hat{1}$ as their unit. The insensitivity of the structures to the topology of their unit then allows a direct and effective representation of nonlocal-integral forces without excessively complex alterations of the original theories.

In particular, systems (1.1) become directly representable via a conventional Hamiltonian H(t, r, p) characterizing all selfadjoint forces, and by embedding all nonhamiltonian forces in the generalized unit $\hat{1}$.

Needless to say, the above representation of systems (1.1) is not unique, and a number of additional possibilities exist in the specialized mathematical literature. However, these methods are rather complex indeed, because requiring rather

delicate nonlocal topologies, and of unknown value for the characterization of a generalized analytic mechanics. The primary value of Santilli's isotopic structures is therefore their simplicity and effectiveness, as we shall see.

The mapping

$$1 \Rightarrow \hat{1}, \tag{1.3}$$

is a first example of isotopies. In fact, the basic properties (or axioms) of the conventional unit 1 are those of being nowhere singular, real valued and symmetric. The lifting $1 \Rightarrow \hat{1}$ is then an isotopy because all the infinite, most general possible isounits (1.2) do preserve these basic properties by assumption. In physical applications Santilli adds the condition of positive-definiteness of the unit and of its isotopic images because it is instrumental in proving the local isomorphism between the isotopic and conventional symmetries.

Santilli fundamental application of the the isotopies $I \Rightarrow \hat{1}$ has been for the construction of a corresponding generalized formulation of Lie's theory which he called *Lie-isotopic theory* (Santilli (*loc. cit.*)), but which is today called *Lie-Santilli theory* (see Aringazin *et al.* (1991), and other papers quoted later on). As we shall see in Sect. 6, the generalized theory essentially consists of isotopic liftings of all the various branches of the conventional Lie's theory (universal enveloping associative algebras, Lie algebras, Lie groups, representation theory, etc.), when formulated with respect to, and under the condition of the existence of the most general possible isounit (1.2).

A few aspects should be indicated in these introductory words. The first is that, owing to the deep inter-relation and mutual compatibility of the various mathematical structures used in dynamics, the isotopies of any one of them require, for mathematical consistency, the isotopies of all the others.

For instance, the isotopies of an algebra soon require, for consistency, those of the underlying field which, in turn, require the isotopies of the space in which their modular actions hold which, in turn, require the isotopies of the applicable geometry, of the transformation theory, etc.

This is the reason why Santilli starts with the isotopies of fields, and then passes to those of linear spaces, metric spaces, algebras, geometries, etc.

A second important aspect of Santilli's analysis, is the restriction of the isotopies to those admitting a well identified (left and right) isounit $\hat{1}$. As well known (see, e.g., Jacobson (1962)), the conventional Lie's theory is formulated with respect to the trivial unit 1 of current use in all its branches. It is then evident that the selection, say, of an isotopy of the associative enveloping algebra which does not possess the unit, even though mathematically relevant, is bound to be inadequate for the quantitative treatment of interior systems of type (1.1).

It is evident that, when the isotopic formulation of a given mathematical structure has been identified, then its study for singular isounits are mathematically and physically relevant. The best example is given by the singularities of the isotopies of the Riemannian geometry due to the isounits, which offer evident new possibilities for gravitational collapse and other topics. Nevertheless, Santilli stresses that regular isotopies should be studied prior to their singular particularization.

A third introductory aspect is that the isotopies essentially represent the "degrees of freedom" of given mathematical axioms and, by central conditions, they produce no new abstract axiomatic structure.

As a matter of fact, this property is so universal that the most effective criterion for ascertaining the mathematical consistency of given isotopies is that *the conventional and isotopic formulations must coincide, by construction, at the abstract, realization-free level*, as stressed since the original proposal (Santilli (1978a)).

As a result, the reader should not expect the identification of new Lie algebras via the use of isotopies, trivially, because all Lie algebras (over a field of characteristic zero) are known from Cartan's classification. On the contrary, *the isotopies produce generally new, infinitely many different, nonlinear and nonlocal realizations of known abstract Lie algebras.*

In fact, Santilli's Lie-isotopic generalizations of Galilei's and Poincaré's symmetries which are at the foundations of his relativities coincide, by conception and realization (for positive-definite isounits), with the conventional Galilei and Poincaré symmetries, respectively. More generally, Santilli's isotopies of Galilei's relativity, Einstein's special relativity and Einstein's general relativity for the interior problem coincide, also by conception and realization, with the conventional relativities of the exterior problem at the level of abstract, realization-free formulations (Santilli (1988a, b, c, d) and (1991a, b)).

In short, the isotopies permit the achievement of a rather remarkable unity of mathematical and physical thought in which, in the transition from the exterior to the interior dynamical problem, the fundamental geometries, space-time symmetries and physical laws, rather than being abandoned, are preserved in their entirety, and only realized in the most general possible nonlinear and nonlocal forms.

The mathematical literature on isotopies is rather limited, to my best knowledge. During his first studies in the topic at the Department of Mathematics of Harvard University[5] in the late 70's, Santilli conducted an extensive search in

[5] Where Santilli had a position as co-principal investigator under contracts with the US Department of Energy Numbers ER-78-S-02-4742.A000, AS02-78ER04742 and DE-AC02-8-

the Cantabridgean mathematical libraries. The *only* mathematical book that he could identify at that time with the notion of isotopy was Bruck (1958), who points out that the notion dates back to the early stages of *set theory*, whereby *two sets were called isotopically related if they could be made to coincide via permutations.*

An extensive search in abstract algebras revealed that the notion had been applied to associative and (commutative) Jordan algebras (see the mathematical bibliography by C. Balzer *et al.* (1984)), but no application of the notion of isotopy to Lie algebras and other structures of direct physical relevance existed at the time of Santilli's original proposal of 1978.

To my best knowledge at this writing (early 1992), Santilli remains the originator and sole author on the mathematical study of the isotopic liftings of Lie algebras, geometries and mechanics; no additional mathematical book has appeared with the notion of isotopy; and the only articles appeared in a mathematical journal with the names "Lie-isotopic algebras" are the review by Aringazin *et al.* (1990), and the two memoirs by Santilli (1991a, b). Quite appropriately, Santilli quotes several times Bruck's (*loc. cit.*) warning:

"*The notion* (of isotopy) *is so natural to creep in unnoticed.*"

I.2: ISOFIELDS

Recall that a *field* (see, e.g., Albert (1963)) is a set F of elements $\alpha, \beta, \gamma, \ldots$ equipped with two (internal) operations, usually called *addition* $\alpha + \beta$ and *multiplication* or *product* $\alpha\beta$, such that

1) Properties of addition: For all $\alpha, \beta, \gamma \in F$, $\alpha + \beta = \beta + \alpha$, and $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$; for each element α there is an element 0, the unit for the addition here called *additive unit*, such that $\alpha + 0 = \alpha$ and an element $-\alpha$ such that $\alpha + (-\alpha) = 0$; and the set is not empty, i.e., there exist elements $\alpha \neq 0$;

2) Properties of multiplication: for all $\alpha, \beta, \gamma \in F$ we have $\alpha\beta = \beta\alpha$ and $\alpha(\beta\gamma) = (\alpha\beta)\gamma$; for all elements $\alpha \in F$ there exists an element 1, the unit for the multiplication here called *multiplicative unit*, such that $\alpha 1 = 1\alpha = \alpha$, and an element α^{-1} such that $\alpha\alpha^{-1} = \alpha^{-1}\alpha = 1$; and the equations $\alpha x = \beta$, and $x\alpha = \beta$, for $\alpha \neq 0$, always admit solution;

3) Distributive laws: for all $\alpha, \beta, \gamma \in F$, $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$, and $(\beta + \gamma)\alpha = \beta\alpha + \gamma\alpha$.

0ER10651 in association with his colleague S. Sternberg.

Unless otherwise stated, all fields are assumed of characteristics zero[6] throughout the analysis of these papers, so as to avoid fields with an axiomatic structure different than that currently used in physics. The extension of the results to fields of characteristic $p \neq 0$ is rather intriguing, but it will be left for brevity to the interested mathematician.

The sets of real numbers $\Re$, complex numbers C and quaternions Q constitute fields with respect to the conventional sum and multiplication. However, octonions O do not constitute a field because of the loss of the associativity of the product

DEFINITION 2.1: Given a field F *with elements* α, β, γ..., *sum* $\alpha + \beta$, *multiplication* $\alpha\beta$, *and respective units* 0 *and* I, *"Santilli's isofields" are rings of elements* $\hat{\alpha} = \alpha\hat{1}$ *where* α *are elements of* F *and* $\hat{1} = T^{-1}$ *is a positive-definite* $n\times n$ *matrix generally outside* F *equipped with the same sum* $\hat{\alpha} + \hat{\beta}$ *of* F *with related additive unit* $\hat{0} = 0$, *and a new multiplication* $\hat{\alpha}*\hat{\beta} = \hat{\alpha}T\hat{\beta}$, under which $\hat{1} = T^{-1}$ *is the new left and right unit of* $\hat{F}$ *in which case* $\hat{F}$ *satisfies all axioms of a field.*

Thus, an isofield is a field by construction. The basic isofields of this analysis are the *real isofields* $\hat{\Re}$, i.e., the infinitely possible isotopes $\hat{\Re}$ of the field of real numbers $\Re$, which can be symbolically written

$$\hat{\Re} = \{\hat{n} \mid \hat{n} = n\hat{1},\ n \in \Re,\ \hat{1} \neq 0\}, \tag{2.1}$$

and their elements $\hat{n}$ are called *isonumbers*. As per Definition 2.1, the sum of two isonumbers is the conventional one,

$$\hat{n}_1 + \hat{n}_2 = (n_1 + n_2)\hat{1}. \tag{2.2}$$

To identify the appropriate isoproduct, recall that $\hat{1}$ must be the right and left isounit of $\hat{\Re}$. This is the case if one interprets $\hat{1}$ as the inverse of an element T, called *isotopic element*,

---

[6] Let F be a field with elements α, β,... If there exists a least positive integer p such that $p\alpha = 0$ for all $\alpha \in F$, then the field F is said to be of *characteristic* p. The fields of real or complex numbers evidently have *characteristic zero*. Contrary to a rather general belief in physical circles, the classification of simple Lie algebras is still incomplete. In fact, we have today Cartan's classification of *all* simple Lie algebras, but *only* over a *field of characteristic zero*, while that over a field of characteristic p is still an open problem at this writing.

$$\hat{1} = T^{-1}, \tag{2.3}$$

and defines the isoproduct as

$$\hat{n}_1 * \hat{n}_2 \stackrel{\text{def}}{=} n_1 T n_2, \quad T \quad \text{fixed.} \tag{2.4}$$

Then,

$$\hat{1} * \hat{n} = \hat{n} * \hat{1} \equiv \hat{n}, \qquad \text{for all } \hat{n} \in \hat{\mathfrak{R}}, \tag{2.5}$$

as desired.

Note that the isotopic element T need not necessarily be an element of the original field $\mathfrak{R}$, because it can be, say, an integro-differential operator. As we shall see, this feature is of fundamental relevance for the applications of the isotopic theory.

Note also that the lifting $I \Rightarrow \hat{1}$ does not imply a change in the numbers used in the practical, mathematical and physical applications. This can be seen in various ways, e.g., from the fact that the isoproduct of an isonumber $\hat{n}$ times a quantity Q coincides with the conventional product,

$$\hat{n} * Q \equiv nQ. \tag{2.6}$$

Note finally, from the complete arbitrariness of the isotopic element T in isoproduct (2.4), that the field of real number $\mathfrak{R}$ admits an infinite number of different isotopies.

Another field of basic physical relevance is the complex isofield $\hat{C}$,

$$\hat{C} = \{\hat{c} \mid \hat{c} = c\hat{1}, \ c \in C, \ \hat{1} \neq 0\} \tag{2.7}$$

which plays a fundamental role in the operator formulation of the classical isotopies of this volume. As such it will be considered elsewhere.

Particularly intriguing is the notion of *quaternionic isofield* $\hat{Q}$ which does not appear to have been studied in the mathematical or physical literature until now, to our best knowledge. In fact, the elements of $\hat{Q}$ are matrices, as well known. Their isotopic liftings via a matrix T of the same dimension then considerably broaden the original structure.

Finally, note that the isotopy $F \Rightarrow \hat{F}$ used in these papers is solely referred to the multiplication, and not to the addition. Needless to say, a more general notion of isotopy including both sum and multiplication as well as internal and external operations is, conceivable, but its study is left to the interested mathematician.

The notion of isofield was submitted by Santilli during an invited talk at the

Clausthal Conference on *Differential Geometric Methods in Mathematical Physics* of 1980 (see also Santilli (1981b)). The notion was then elaborated by Myung and by Santilli (1982a), and by various works by Santilli (1983a), (1985a), (1988a, b) and (1991a, b).

I.3: ISOSPACES

A *linear space* V (see again Albert (1963)) is a set of elements a,b,c,... over a field F of elements α, β, γ, ... and units 0 and I, equipped with the additions a+b, and a + b, and the multiplications ab, aa, and ab, such that, for all a,b,c $\in$ V and α, β,γ $\in$ F: $a + b = b + a$; $a + (b + c) = (a + b) + c$; $\alpha(\beta a) = (\alpha\beta)a$; $\alpha(a + b) = \alpha a + \alpha b$; $(a + b)\alpha = \alpha a + \beta a$; for every a $\in$ V there exists an element $-a$ such that $a + (-a) = a - a = 0$; and the multiplicative unit I of F is the right and left unit of V, i.e., $Ia = aI = I$ for all a $\in$ V.

From the above definition one can clearly see that we cannot construct an isotopy of a linear space without first introducing an isotopy of the field, because the multiplicative unit I of the space is that of the underlying field.

DEFINITION 3.1: Given a linear space V over a field F, the "isotope" $\hat{V}$ *of V with respect to the multiplication, here called "Santilli's isospace", is the same set of elements a, b, c,...* $\in$ *V defined over the isofield* $\hat{F}$ *with multiplicative isounit* $\hat{1}$ *and therefore equipped with a new multiplication a*b, which is such to verify all the axioms for a linear space, i.e.,*

$$\alpha*(\beta*a) = (\alpha*\beta)*a, \quad \alpha*(a + b) = \alpha*a + \alpha*b, \tag{3.1a}$$

$$(\alpha + \beta)*a = \alpha*a + \beta*a, \; \alpha*(a + b) = \alpha*a + \alpha*b, \tag{3.1b}$$

$$\hat{1}*a = a*\hat{1} \equiv a, \tag{3.1c}$$

for all a,b $\in$ V *and* $\alpha, \beta \in$ F.

Note the lifting of the field, but the elements of the vector space remain unchanged. This is a property of important physical consequence, inasmuch as it is at the foundation of the preservation of the conventional generators of Lie algebras under isotopies. In turn, this implies the preservation of conventional conservation laws under lifting.

The interested reader can prove as an exercise a number of properties of isolinear spaces. One which is particularly relevant for this analysis follows from the invariance of the elements a, b, c, ... of the space under isotopy and can be

expressed as follows.

PROPOSITION 3.1: The basis of a (finite-dimensional) linear space V remains unchanged under isotopy.

The above property essentially anticipates the fact that, when studying later on the Lie-Santilli algebras, we shall expect no alteration of its basis (Santilli (1978a)).

Linear spaces V are also called *vector spaces* in which case their elements a,b,c, are called *vectors*. The isotopes $\hat{V}$ are then called *isovector spaces* and their elements a, b, c *isovectors*.

A *metric space* hereon denoted M(x,g,F) is a (universal) set of elements x, y, z,.. over a field F equipped with a map (function) g: M x M ⇒ F, such that:

$$g(x,y) \gtreqless 0, \tag{3.2a}$$

$$g(x,y) = g(y,x) \text{ for all } x,y \in M; \qquad g(x,y) = 0 \text{ iff } a = 0 \text{ or } b = 0. \tag{3.2b}$$

$$g(x,y) \leqq g(x,z) + g(y,z) \text{ for all } x,y,z \in M. \tag{3.2c}$$

A *pseudo-metric space*, hereon also denoted with M(x,g,F), occurs when the first condition (3.2a) is removed. Finally, recall that the field of metric spaces generally used in physics is that of the reals $\Re$.

Suppose that the space M(x,g,) is n-dimensional, and introduce the components $x = (x^i)$, $y = (y^i)$, $i = 1, 2, ..., n$. Then, the familiar way of realizing the map g(x,y) is that via a *metric* g of the form

$$g(x,y) = x^i g_{ij} y^j, \tag{3.3}$$

The axiom g(x,y) > 0 for metric spaces then implies that g is positive-definite, g > 0.

The best physical example of a metric space is the n-dimensional *Euclidean space* hereon denoted with the symbol E(r,δ,$\Re$), namely, the vector space E with local charts $r = (r^i)$ and realization of the metric

$$g(r_1,r_2) = r_1{}^i \delta_{ij} r_2{}^j, \tag{3.4}$$

where

$$\delta = (\delta_{ij}) = \mathrm{diag.}\,(1, 1, ..., 1) \tag{3.5}$$

is the matrix of the *Kronecker delta* δ_{ij}.

A pseudo-metric space of primary physical relevance is the (3+1)-dimensional *Minkowski space* hereon denoted $M(x,\eta,\Re)$, namely, the vector space with charts

$$x = (x^\mu) = (x^i, x^4), \quad x^i \in E(r,\delta,\Re), \quad x^4 = c_0 t, \tag{3.6}$$

where $c_0 \in \Re$ represents the speed of light in vacuum. Then the map is indefinite,

$$\eta(x,y) = x^\mu \eta_{\mu\nu} y^\nu \;\substack{> \\ = \\ <}\; 0, \tag{3.7}$$

where η is the celebrated Minkowski (1913) metric, hereon assumed of the type

$$\eta = \mathrm{diag.}\,(1, 1, 1, -1). \tag{3.8}$$

Further spaces also relevant in physics are the *Riemannian spaces* hereon denoted $R(x,g,\Re)$, which are the fundamental spaces of Sect.s 11 and 12 of Part II.

The simplest possible way of constructing an infinite family of isotopes of $M(x,g,F)$ is by introducing n-dimensional, nowhere null and Hermitean isounits

$$\hat{1} = (\hat{1}_i^{\,j}) = (\hat{1}^j_{\,i}), \quad i, j, r, s, = 1, 2, ..., n. \tag{3.9}$$

with isotopic elements

$$T = \hat{1}^{-1} = (T_i^{\,j}) = (T^j_{\,i}), \tag{3.10}$$

Then, we can introduce the notion of the *isomap*

$$\hat{g}(x,y) = (x^i\, \hat{g}_{ij}\, y^j)\, \hat{1}, \tag{3.11}$$

where the quantity

$$\hat{g} = Tg = (T_i^{\,k}\, g_{kj}), \tag{3.12}$$

shall be called hereon the *isometric*.

The *basis* $e = (e_i)$, $i = 1, 2, \dots, n$ of an n-dimensional space $M(x,g,F)$ can be defined via the rules

$$g(e_i, e_j) = g_{ij}. \tag{3.13}$$

Then, under isotopy we have the rules

$$\hat{g}(e_i, e_j) = \hat{g}_{ij}, \tag{3.14}$$

which illustrate the preservation of the basis as per Proposition 3.1.

The above isotopic generalizations can be expressed as follows.

DEFINITION 3.2: The "isotopic liftings" of a given, n-dimensional, metric or pseudometric space $M(x,g,\Re)$ *are given by the infinitely possible Santilli's isospaces* $\hat{M}(x,\hat{g},\hat{\Re})$ *characterized by: a) the same dimension n and the same local coordinates x of the original space; b) the isotopies of the original metric g into one of the infinitely possible nonsingular, Hermitean "isometric"* $\hat{g} = Tg$ *with isotopic element T depending on the local variables x, their derivatives* $\dot{x}$, $\ddot{x}$, ... *with respect to an independent parameter, as well as any needed additional quantity*

$$g \Rightarrow \hat{g} = Tg, \tag{3.15a}$$

$$T = T(x, \dot{x}, \ddot{x}, \dots), \quad \det T \neq 0, \quad T^{\dagger} = T, \quad \det. g \neq 0, \quad g = g^{\dagger}, \tag{3.15b}$$

and c) the lifting the field $\Re$ *into an isotope* $\hat{\Re}$ *whose isounit* $\hat{1}$ *is the inverse of the isotopic element T, i.e.,*

$$\hat{\Re} = \Re\hat{1}, \quad \hat{1} = T^{-1} = \delta^{-1}, \tag{3.16}$$

with composition now over $\hat{\Re}$

$$(x \,\hat{,}\, y) = (x, Ty)\hat{1} = (Tx, y)\hat{1} = \hat{1}(x, Ty)$$
$$= (x^i \hat{g}_{ij} y^j)\hat{1} \in \hat{F}, \tag{3.17}$$

The liftings of the conventional n-dimensional Euclidean spaces $E(r,\delta,\Re)$ *over the*

reals $\Re$ into "Santilli's isoeuclidean spaces", are given by the particular case

$$E(r,\delta,\Re) \Rightarrow \hat{E}(r,\hat{\delta},\hat{\Re}), \tag{3.18a}$$

$$\delta = I_{n\times n} \Rightarrow \hat{\delta} = T(r, \dot{r}, \ddot{r}, ...)\,\delta, \tag{3.18b}$$

$$\det \delta = 1 \neq 0,\ \delta = \delta^{\dagger} \Rightarrow \det. \hat{\delta} \neq 0,\ \ \hat{\delta} = \hat{\delta}^{\dagger}, \tag{3.18c}$$

$$\Re \Rightarrow \hat{\Re} = \Re\hat{1},\ \ \hat{1} = T^{-1} = \hat{\delta}^{-1} \tag{3.18d}$$

$$(r,r) = r^{i}\,\delta_{ij}\,r^{j} \Rightarrow (r\,\hat{,}\,r) = (r,\hat{\delta}r)\,\hat{1} \tag{3.18e}$$

$$= (\hat{\delta}r,r)\,\hat{1} = \hat{1}\,(r,\hat{\delta}r) = [r^{i}\,\hat{\delta}_{ij}(r, \dot{r}, \ddot{r},..)\,r^{j}]\,\hat{1}, \tag{3.18f}$$

The liftings of the conventional Minkowski space $M(x,\eta,\Re)$ *in (3+1)-space-time dimensions are given by the isotopes called "Santilli's isominkowski spaces"*

$$M(x,\eta,\Re) \Rightarrow \hat{M}(x,\hat{\eta},\hat{\Re}), \tag{3.19a}$$

$$\eta = \mathrm{diag.}(1, 1, 1, -1) \Rightarrow \hat{\eta} = T(x, \dot{x}, \ddot{x},...)\,\eta, \tag{3.19b}$$

$$\det \eta = -1 \neq 0,\ \ \eta = \eta^{\dagger} \Rightarrow \det \hat{\eta} \neq 0,\ \ \hat{\eta}^{\dagger} = \hat{\eta}, \tag{3.19c}$$

$$\Re \Rightarrow \hat{\Re} = \Re\hat{1},\ \ \hat{1} = T^{-1}, \tag{3.19d}$$

$$(x,x) = x^{\mu}\,\eta_{\mu\nu}\,x^{\nu} \Rightarrow (x\,\hat{,}\,x) = (x,Tx)\,\hat{1} = (Tx,y)\,\hat{1}$$
$$= \hat{1}(x,Ty) = [x^{\mu}\,\hat{\eta}_{\mu\nu}(x, \dot{x}, \ddot{x},...)\,x^{\nu}\,]\,\hat{1}, \tag{3.19e}$$

Finally, the liftings of a given n-dimensional, Riemannian or pseudoriemannian space $R(x,g,\Re)$ *over the reals $\Re$ into the infinitely possible isotopes called "Santilli's isoriemannian spaces"* $\hat{R}(x,g,\hat{\Re})$ *are given by*

$$R(x,g,\Re) \Rightarrow \hat{R}(x,\hat{g},\hat{\Re}), \tag{3.20a}$$

$$g = g(x) \Rightarrow \hat{g} = T(x, \dot{x}, \ddot{x}, ...)\,g(x), \tag{3.20b}$$

$$\det. g \neq 0,\ \ g = g^{\dagger} \Rightarrow \det. \hat{g} \neq 0,\ \ \hat{g} = \hat{g}^{\dagger}, \tag{3.20c}$$

$$\Re \Rightarrow \hat{\Re} = \Re \hat{1}, \quad \hat{1} = T^{-1}, \tag{3.20d}$$

$$(x,y) = x^i g_{ij}(x) x^j \Rightarrow (x \hat{,} x) = (x,Tx)\hat{1} = (Tx,x)\hat{1}$$

$$= \hat{1}(x,Tx) = (x^i \hat{g}_{ij}(x, \dot{x}, \ddot{x}, ...) x^j]\hat{1}. \tag{3.20e}$$

Santilli)1988b) illustrated the general character of the concept of isotopy via the following property of evident proof.

PROPOSITION 3.2: All possible metric and pseudometric spaces in n-dimension $M(r,g,\Re)$ *can be interpreted as isotopes of the Euclidean space in the same dimension* $E(r,\delta\Re)$ *under the reformulation*

$$\hat{M}(r,g,\hat{F}): \qquad \hat{F} = F\hat{1}, \; \hat{1} = g^{-1}. \tag{3.21}$$

The reader should therefore keep in mind that there is no need to study the isotopies of all spaces, because those of the fundamental Euclidean space are sufficient, and inclusive of all others, as illustrated by Santilli (*loc. cit.*) with the following

COROLLARY: 3.2a: The conventional Minkowski space $M(x,\eta,\Re)$ *in (3+1) space-time dimensions over the reals* $\Re$ *can be interpreted as an isotope* $\hat{M}(r,\eta,\hat{\Re})$ *of the 4-dimensional Euclidean space* $E(x,\delta,\Re)$ *characterized by the isotopy of the metric*

$$\delta = I_{4x4} = \hat{\delta} = T\delta = \eta = \text{diag.}(1, 1, 1, -1), \tag{3.22}$$

under the redefinition of the fields

$$\Re \Rightarrow \hat{\Re} = \Re\hat{1}, \quad \hat{1} = T^{-1} = \eta^{-1} = \eta. \tag{3.23}$$

The reader should remember that the isotopy of the field is a feature needed for mathematical consistency, but it does not affect the practical numbers of the theory. Also, as we shall see in Sect. 8, the symmetries of $\hat{M}(x,\eta,\hat{\Re})$ and those of $M(x,\eta,\Re)$ coincide because characterized by the metric η. Thus, the isotopic Minkowski space $\hat{M}x,\eta,\hat{\Re})$ and the conventional Minkowski space $M(x,\eta,\Re)$ can be

made to coincide for all practical purposes used in physics.

COROLLARY 3.2.b: The conventional Riemannian spaces R(x,g,$\Re$) *in (3+1)-space-time dimensions over the reals $\Re$ is an isotope $\hat{R}(x,g,\hat{\Re})$ of the 4-dimensional Euclidean space* E(x,δ,$\Re$) *characterized by the lifting of the Euclidean metric δ into the Riemannian metric* g

$$\delta = 1_{4\times 4} \Rightarrow T\delta = g, \tag{3.24}$$

and by the corresponding lifting of the field

$$\Re \Rightarrow \hat{\Re} = \Re\hat{1}, \quad \hat{1} = T^{-1} = g^{-1}. \tag{3.25}$$

Santilli (*loc. cit.*) also submitted the following alternative interpretation of the Riemannian space.

COROLLARY 3.2.c: The conventional Riemannian space R(x,g,$\Re$) *in (3+1)-space-time dimensions over the reals $\Re$ can be interpreted as an isotope* $\hat{R}(x,g,\hat{\Re})$ *of the Minkowski space* M(x,η,$\Re$) *in the same dimension characterized by the isotopy of the Minkowski metric*

$$\eta = \mathrm{diag.}\,(1, 1, 1, -1) \Rightarrow T(x)\,\eta = g(x), \tag{3.26}$$

and of the field

$$\Re \Rightarrow \hat{\Re} = \Re\hat{1}, \quad \hat{1} = T^{-1}. \tag{3.27}$$

The notion of isotopy of a metric or pseudometric space is therefore first useful for *conventional* formulations. In fact, Santilli (*loc. cit.*) has shown that *the transition from relativistic to gravitational aspects is an isotopy.* This concept is at the foundations of his study of the *global symmetries of conventional gravitational theories* which can be readily characterized by the Lie-Santilli theory, but which are otherwise of rather difficult treatment via conventional techniques.

Notice also the *chain of isotopies* illustrated by the above Corollaries, also called *isotopies of isotopies,*

$$E(x,\delta,\Re) \Rightarrow M(x,\eta,\Re) \Rightarrow R(x,g,\Re). \tag{3.28}$$

Corollary 3.2.c is useful to illustrate the insensitivity of the isotopies to the

explicit functional dependence of the isounit. The reader can then begin to see the general character of Santilli's isoeuclidean spaces which encompass, not only the Minkowski and Riemannian spaces, but also all known metric and pseudometric spaces of the same dimension, such as Finslerian spaces, as well as additional classes of infinitely possible, genuine isotopies of the Euclidean, Minkowski, Riemannian and other spaces.

DEFINITION 3.3: Given a metric or pseudo-metric space M(x,g,F) *with metric* g, *"Santilli's isodual" space* $\hat{M}^d(x,\hat{g},\hat{F})$ *is the isotopic space* $\hat{M}$ *characterized by the isotopic element*

$$T = -1 = \text{diag.}(-1, -1, -1, ..., -1). \tag{3.29}$$

The isodual of the Euclidean space $E(x,\delta,\Re)$ is therefore the isotope $\hat{E}^d(x,\hat{\delta},\hat{\Re})$ where the isometric is given by

$$\hat{\delta} = -\delta. \tag{3.30}$$

As we shall see, the above spaces are useful for the construction of the isodual realization of given simple Lie groups with rather intriguing implications, evidently given by the embedding of the inversion in the isounit of the theory.

Similarly, the isodual of the Minkowski space $M(x,\eta,\Re)$ is the isospace $\hat{M}^d(x,\hat{\eta},\hat{\Re})$ where the isometric $\hat{\eta}$ is given by

$$\hat{\eta} = T\eta = -\eta = \text{diag.}(-1, -1, -1, +1). \tag{3.31}$$

Clearly the notion of isoduality in Minkowski space allows the mapping of time-like into space-like vectors and vice-versa. As such, isodual spaces are at the basis of the generalized Lorentz transformations $x \Rightarrow x'(x)$ introduced by Recami and Mignani (1972) for which

$$x'^{\mu}\, \eta_{\mu\nu}\, x'^{\nu} = -\, x^{\mu}\, \eta_{\mu\nu}\, x^{\nu}, \tag{3.32}$$

and are important to identify certain properties of the isotopies of the Lorentz group (Santilli (1991b)).

The notion of isospace was introduced by Santilli (1983a) in conjunction with his first construction of the isotopic generalization of Einstein's special relativity. The theory of isospaces was then used in more details for the construction of the infinite isotopies of isometries of metric and pseudometric spaces (Santilli (1985a,

b)), where the notion of isotopic duality was also introduced. Finally, the extension of the theory to Riemannian spaces was done recently in Santilli ((1988d), (1991a,b)).

The infinite family of isotopic generalizations of the Euclidean space, of the Minkowski space and of the Riemannian space, constitute the foundations of Santilli's isotopic generalization of Galilei's relativity, Einstein's special relativity, and Einstein's gravitation, respectively.

The reason why they are infinite in number is to be able to represent the infinitely possible interior conditions for each given exterior gravitational mass.

Also, the reader can see that Santilli's isoeuclidean, isominkowski and isoriemannian spaces provide a form of "geometrization" of the infinitely possible interior physical media when studied from a nonrelativistic, relativistic and gravitational profile, respectively. For numerous physical applications along these lines, we suggest the consultation of Santilli (1991c, d).

I.4: ISOTRANSFORMATIONS

Let V and V′ be two linear spaces over the same field F. A *linear transformation* (Albert (1963)) is a map f: V ⇒ V′ which preserves both the sum and the multiplication, i.e., it is such that

$$f(a + b) = f(a) + f(b), \tag{4.1a}$$

$$f(\alpha a) = f(\alpha)f(a), \tag{4.1b}$$

which can be equivalently written

$$f(\alpha a + \beta b) = f(\alpha)f(a) + f(\beta)f(b) \text{ for all } a,b \in V \text{ and } \alpha, \beta \in F. \tag{4.2}$$

DEFINITION 4.1: "Santilli's isotopic transformations" are isomaps $\hat{f}$: $\hat{V} \Rightarrow \hat{V}'$ among two isolinear vector spaces $\hat{V}$ and $\hat{V}'$ of the same dimension over the same isofield $\hat{F}$ which preserves the sum and isomultiplication, i.e., which are such that

$$\hat{f}(\alpha * a + \beta * b) = \hat{f}(\alpha) * \hat{f}(a) + \hat{f}(\beta) * \hat{f}(b) \text{ for all } a,b, \in V \text{ and } \alpha, \beta \in \Re \tag{4.3}$$

In physical applications, the spaces V and V′ are usually assumed to coincide, V ≡ V′, in which case the *linear map* f is an *endomorphism* with realizations of the familiar right, modular-associative type

$$x' = Ax, \quad x \in V, \quad x' \in V'. \tag{4.4}$$

where: A is independent from the local variables; the product Ax is associative; and the notion of module will be treated in more details in the next section. A similar notion would evidently result for a left modular associative action $x' = xA$.

The transformations are *nonlinear* when of the form

$$x' = A(x)\, x. \tag{4.5}$$

i.e., when A has an explicit dependence in the local coordinates x. If the x–dependence is of integral type, we shall say that the above transformations are nonlocal.

Assume now that $\hat{V} \equiv \hat{V}'$. Then the isomap $\hat{T}$ can be realized with the isotransformations characterized by the right modular, associative isotopic action

$$x' = A{*}x = ATx, \quad T = \text{fixed}. \tag{4.6}$$

where the action $A{*}a$ is still associative. A similar notion would result for a left, modular–isotopic action $x' = x{*}A = xTA$.

DEFINITION 4.2: Santilli's isotransformations (4.6) are said to be "isolinear" and/or "isolocal" when the element A is conventionally linear and/or local, respectively, i.e., when all nonlinear and/or nonlocal terms are embedded in the isotopic element T.

A number of properties of isotransformations can be easily proved. At the level of abstract axioms, all distinctions between the ordinary multiplication ab and the isotopic one $a{*}b$ (transformations Ax and $A{*}x$) cease to exist, in which case linear and isolinear spaces (linear and isolinear transformations) coincide.

However, the isotopies are nontrivial, as illustrated by a number of properties. First, Santilli (1988b) points out the following

PROPOSITION 4.1: Conventional linear transformations f *on an isolinear space* $\hat{V}$ *violate the conditions of isolinearity.*

Explicitly stated, the lifting of the Euclidean spaces and of the Minkowski spaces into their corresponding isospaces requires the necessary abandonment, for mathematical consistency, of the Galilean and Lorentz transformations in favor of Santilli's isolinear and isolocal generalizations.

A most important property of Santilli's isotransformations is given by the following

PROPOSITION 4.2: A transformation $\hat{f}$ which is isolinear and isolocal in an isospace $\hat{V}$ is generally nonlinear and nonlocal in V.
In fact, when explicitly written out, isotransformations (4.6) become

$$x' = ATx = A\,T(t, x, \dot{x}, \ddot{x}, \ldots)\, x. \tag{4.7}$$

the nonlinearity and nonlocality of the transformations then becomes evidently dependent on the assumed explicitly form of the isotopic element T.

Another simple but important property is the following

PROPOSITION 4.3: Under sufficient topological conditions, nonlinear transformations on a linear vector V space can always be cast into an equivalent isolinear form on an isospace $\hat{V}$.

In different terms, given a map f in V which violates the conditions of linearity and/or of locality, there always exist an isotope $\hat{V}$ of V under which $\hat{f}$ is isolinear and/or isolocal. Explicitly, nonlinear transformations (4.5) can always be written

$$x' = A(x)\, x \equiv BT(x)x = B{*}x, \tag{4.8}$$

i.e., for A = BT, with B linear.

The above property has important mathematical and physical implications. On mathematical grounds we learn that *nonlinearity and nonlocality are mathematical characteristics without an essential axiomatic structure, because they can be made to disappear at the abstract level via isotopic liftings.*

In turn, this feature is not a mere mathematical curiosity, but has a number of possible mathematical applications. As an example, if properly developed, the isotopies of the current theory of linear equations may be of assistance in solving equivalent nonlinear systems.

On physical grounds, the first application of the notions presented in this section is that of rendering more manageable the formulation and treatment of nonlinear and nonlocal generalizations of Galilean or Lorentzian theories which, if treated conventionally, are of a notoriously difficult (if not impossible) treatment.

The physical implications are however deeper than that. Recall that the electromagnetic interactions have been fully treatable with linear and local theories, such as the symmetry under the conventional Lorentz transformations.

One of the central open problems of contemporary theoretical physics (as well as of applied mathematics) is the still unanswered, historical legacy by Fermi (1949) and other Founders of contemporary physics on the ultimate nonlinearity and nonlocality of the strong interactions.

All attempts conducted until now in achieving a nonlinear and nonlocal extension of current theories via conventional techniques have met with rather serious problem of mathematical consistency and/or physical effectiveness, as well known.

Because of their simplicity, Santilli's isotopies appear to have all the necessary ingredients for the achievement of a mathematically consistent and physically effective nonlinear and nonlocal generalization of the current theories for the electromagnetic interactions via the mere generalization of their trivial unity I into Santilli's isounit $\hat{1}$, and the consequential isotopic generalization of the various notions of field, spaces, transformations, etc.

The mathematical consistency of the isotopies is self-evident from their simplicity. Their physical effectiveness is due to the fact that, given a linear theory, say a Hamiltonian description of a conservative trajectory on a metric space, all the possible nonlinear and nonlocal generalizations are guaranteed by the mere isotopies of the underlying space.

The (one-sided) isotransformation theory reviewed in this section was originally submitted by Santilli as a particular case of a still more general, two-sided, left and right isotransformation theory for Lie-admissible algebras (Santilli (1979)) (see Appendix D for a review). The isotransformation theory was then studied in detail in the monograph (Santilli (1981a)) and became a second central tool, following the notion of isominkowski space, for the first construction of the isotopies of the special relativity (Santilli (1983a)). Additional relevant studies were conducted in Myung and Santilli (1982a), Mignani, Myung and Santilli (1983), and Santilli (1982a), (1988a, b), (1991a, b, c, d).

I.5: ISOALGEBRAS

A (finite-dimensional) *linear algebra* U, or *algebra* for short (see, e.g., Albert (1963) or Oehmke *et al.* (1974)) is a linear vector space V over a field F equipped with a multiplication ab verifying the following axioms

$$\alpha(ab) = (\alpha a)b = a(\alpha), \quad (ab)\beta = a(b\beta) = (a\beta)b, \tag{5.1a}$$

$$a(b + c) = ab + ac, \quad (a + b)c = ac + bc, \tag{5.1b}$$

called *right and left scalar and distributive laws,* respectively, which must hold for all elements $a,b,c \in U$, and $\alpha, \beta \in F$.

The reader should keep in mind that *the above axioms must be verified by all products to characterize an algebra* (see Appendix A of Part II for products commonly used in physics which do not characterize a consistent algebra).

Santilli (1988b), (1991a) stresses that algebras play a fundamental role in physics, and their use is predictably enlarged by the isotopies. Among the existing large number of algebras, a true understanding of the isotopic relativities at the classical and/or at the operator level requires a knowledge of the following primary algebras (see, e.g., Albert (1963) and Schafer (1966)):

1) Associative algebras A, characterized by the additional axiom (besides laws (5.1))

$$a(bc) = (ab)c \tag{5.2}$$

for all $a,b,c \in A$, called the *associative law.* Algebras violating the above law are called *nonassociative.* All the following algebras are nonassociative:

2) Lie algebras L which are characterized by the additional axioms

$$ab + ba = 0, \tag{5.3a}$$

$$a(bc) + b(ca) + (c(ab) = 0. \tag{5.3b}$$

A familiar realization of the Lie product is given by

$$[a,b]_A = ab - ba, \tag{5.4}$$

with the classical counterpart being given by the familiar Poisson brackets among functions A, B in phase space $T^*E(r,\delta,\Re)$ (or the cotangent bundle of Sect. 9, Part II)

$$[A,B]_{\text{Poisson}} = \frac{\partial A}{\partial r^k}\frac{\partial B}{\partial p_k} - \frac{\partial B}{\partial r^k}\frac{\partial A}{\partial p_k}. \tag{5.5}$$

3) Commutative Jordan algebras J, characterized by the additional axioms

$$ab - ba = 0, \tag{5.6a}$$

$$(ab)a^2 = a(ba^2), \tag{5.6b}$$

A realization of the special commutative Jordan product is given by

$$(a,b)_A = ab + ba. \tag{5.7}$$

No realization of the commutative Jordan product in classical mechanics is known at this writing. As an example, the brackets

$$(A,B) = \frac{\partial A}{\partial r^k}\frac{\partial B}{\partial p_k} + \frac{\partial B}{\partial r^k}\frac{\partial A}{\partial p_k}, \tag{5.8a}$$

evidently verify axiom (5.6a), but violate axiom (5.6b).

4) <u>General Lie-admissible algebras</u> U (Albert (1948), Santilli (1967), (1968) and (1978a)), which are characterized by a product ab verifying laws (5.1), which is such that the attached product $[a,b]_U$ = ab − ba is Lie. This implies, besides (5.1), the unique axiom

$$(a,b,c) + (b,c,a) + (c,a,b) = (c,b,a) + (b,a,c) + (a,c,b), \tag{5.9}$$

where

$$(a,b,c) = a(bc) - (ab)c, \tag{5.10}$$

is called the *associator.*

Note that *Lie algebras are a particular case of the Lie-admissible algebras.* In fact, given an algebra L with product ab = $[a,b]_A$, the attached algebra L^- has the product $[a,b]_U = 2\,[a,b]_A$ and, thus, L is Lie-admissible.

Therefore, the classification of the Lie Lie-admissible algebras contains all possible Lie algebras. Also, Lie algebras enter in the Lie-admissible algebras in a two-fold way: first, in their classification and, second, as the attached antisymmetric algebras. Finally, *associative algebras are trivially Lie-admissible.*

The first abstract realization of the general Lie-admissible algebras was given by Santilli ((1978b), Sect. 4.14) and can be written

$$U: \quad (a,b)_A = arb - bsa, \tag{5.11}$$

$$r,s \ \text{fixed} \in A, \quad r \neq s, \quad r,s \neq 0$$

where ar, rb, etc., are associative. In fact, the antisymmetric product attached to U

is a particular form of a Lie algebra (see below).

The first realization of U in classical mechanics was also identified by Santilli (1969) and (1978a) and it is given by the following product for functions A(r,p) and B(r,p) in T*E(r,δ,ℜ)

$$U: \quad (A,B) = \frac{\partial A}{\partial r^k} \frac{\partial B}{\partial p_k}, \tag{5.12}$$

namely, *the general, nonassociative Lie-admissible algebras are at the foundations of the structure of the conventional Poisson brackets,* which can be written

$$[A,B]_{\text{Poisson}} = [A,B]_U = (A,B) - (B,A), \tag{5.13}$$

5) *Flexible Lie-admissible algebras* U (Albert (1948), Santilli (1967), (1968) and (1978a)), which are characterized by the axioms in addition to (5.1)

$$(a,b,a) = 0, \tag{5.14a}$$

$$(a,b,c) + (b,c,a) + (c,a,b) = 0, \tag{5.14b}$$

where condition (5.14a), called the *flexibility law,* is a simple generalization of the anticommutative law, as well as a weaker form of associativity. An abstract realization of the flexible Lie-admissible product is given by (Santilli (1978b))

$$(a,b)_U = \lambda\, ab - \mu\, ba, \quad \lambda, \mu \in F \tag{5.15}$$

where the products λa, ab, etc. are associative. No classical realization of flexible Lie-admissible algebras has been identified until now, to the best knowledge of this author. As an example, the brackets on T*E(r,δ,ℜ)

$$(A,B) = \lambda \frac{\partial A}{\partial r^k} \frac{\partial B}{\partial p_k} - \mu \frac{\partial B}{\partial r^k} \frac{\partial A}{\partial p_k} \tag{5.16}$$

are Lie-admissible, but violate the flexibility law.

6) *General Jordan-admissible algebras* U (Albert (1948), Santilli (1978a, b)), which are characterized by a product ab verifying laws (5.1), such that the attached symmetric product $\{a,b\}_U = ab + ba$ is Jordan, i.e., verifies the axiom

$$(a^2,b,a) + (a,b,a^2) + (b,a^2,a) + (a,a^2,b) = 0. \tag{5.17}$$

Again, *associative and Jordan algebras are trivially Jordan- admissible.* Also, Jordan algebras enter in the Jordan-admissible algebras in a two-fold way, in the classification of the latter, as well as the attached symmetric algebras.

It is important for the operator formulation of the isotopies of this volume to point out that Lie-admissible product (5.11) is, jointly, Lie-admissible and Jordan-admissible (Santilli (1978b)), because the attached symmetric product characterizes a special commutative Jordan algebra (see below).

Finally, we should note that the classical Lie-admissible product (5.12) is only Lie-admissible and not jointly Jordan-admissible.

7) *Flexible Jordan-admissible algebras* U (Albert (1948), Santilli (1978a, b)), which, in addition to axioms (5.1), are characterized by the axioms

$$a(ba) = (ab)a, \tag{5.18a}$$

$$a^2(ba) + a^2(ab) = (a^2b)a + (a^2a)b. \tag{5.18b}$$

The flexible Lie-admissible product (5.15) is also a flexible Jordan-admissible product, but the classical product (5.16) is only Lie-admissible, and not flexible Lie-admissible nor Jordan-admissible.

We now pass to the study of the isotopies of the above notions.

*DEFINITION 5.1 (Santilli (1978a)): An "isoalgebra", or simply an "isotope" $\hat{U}$ of an algebra U with elements a,b,c,... and product ab over a field F, is the same vector space U but defined over the isofield $\hat{F}$, equipped with a new product a*b, called "isotopic product", which is such to verify the original axioms of U.*

Thus, by definition, the isotopic lifting of an algebra does not alter the type of algebra considered.

It is important for this monograph (as well as for its operator formulation) to review the isotopies of the primary algebras listed above, beginning with the associative algebras.

Given an associative algebra A with product ab over a field F, its simplest possible *isotope* $\hat{A}$, called *associative-isotopic* or *isoassociative algebra*, is given by

$$\hat{A}_1 : \quad a*b = \alpha ab, \qquad \alpha \in F, \text{ fixed and } \neq 0, \tag{5.19}$$

and called a *scalar isotopy*. The preservation of the original associativity is trivial

in this case.

A second less trivial isotopy is the fundamental one of the Lie-isotopic theory, and it is characterized by product (Santilli (*loc. cit.*))

$$\hat{A}_2 : \quad a*b = aTb, \tag{5.20}$$

where T is an nonsingular (invertible) and Hermitean elements not necessarily belonging to the original algebra A.

Note the necessary condition, from Definition 5.1, that the isoproduct and isounit in U and F coincide. This is the technical reason for the lifting of the universal enveloping associative algebras of a Lie algebra (Sect. 6) into a form whose center coincides with the isounit of the underlying isofield.

The reader should keep in mind that the identity of the isoproduct and isounit for U and F occurs in the associative cases (5.19) and (5.20), but does not hold in general, e.g., for nonassociative algebras. This is due to their lack of general admittance of a unit, while such a unit is always well defined in the underlying field.

Only a third significant isotopy of an associative algebra is known, to the author's best knowledge. It is given by (Santilli (1980), (1981b))

$$\hat{A}_3 : \quad a*b = wawbw, \qquad w^2 = ww = w \neq 0, \tag{5.21}$$

Additional isotopies are given by the combinations of the preceding ones, such as

$$\hat{A}_4 : \quad a*b = wawTwbw, \qquad w^2 = ww = w \neq 0 \tag{5.22}$$

and

$$\hat{A}_5 : \; a*b = \alpha wawTwbw, \qquad \alpha \in F, \quad w^2 = w, \quad a, w, T \neq 0. \tag{5.23}$$

It is believed that the above isotopies (of which only the first three are independent) exhaust all possible isotopies of an associative algebra, although this property has not been rigorously proved to this writing.

The issue is not trivial, physically and mathematically. In fact, any new isotopy of an associative algebra implies a potentially new mechanics, while having intriguing mathematical implications (see later on Lemma 5.1).

It should be finally indicated that Santilli has selected isotopy (5.20) over

(5.21) because the former possesses a well defined isounit, while the latter does not, thus creating a host of problems of physical consistency in its possible use for an operator theory.

Nevertheless, the study of isoassociative algebras (5.21) remains intriguing indeed, although it has not yet been conducted in the mathematical and physical literature, to our best knowledge.

We now pass to the study of the *isotopes* $\hat{L}$ of a Lie algebra L with product ab over a field F, which are the same vector space L but equipped with a *Lie-Santilli product* (Santilli (1978a, c)) a⊙b over the isofield $\hat{F}$ which verifies the left and right scalar and distributive laws (5.1), and the axioms

$$a \odot b + b \odot a = 0, \tag{5.24a}$$

$$a \odot (b \odot c) + b \odot (c \odot a) + c \odot (a \odot b) = 0, \tag{5.24b}$$

Namely, the abstract axioms of the Lie algebras remain the same by assumption.

The simplest possible *realization of the Lie-Santilli product* is that attached to isotopes $\hat{A}_1$, Eq. (5.19)

$$\hat{L}_1: \quad (a,b)_{\hat{A}_1} = a \odot b - b \odot a = \alpha(ab - ba) = \alpha\,[a,b]_A, \qquad \alpha \in F, \quad \alpha \neq 0 \tag{5.25}$$

and it is also called a *scalar isotopy*. It is generally the first lifting of Lie algebras one can encounter in the operator formulation of the theory.

The second independent realization of the Lie-isotopic algebras is that characterized by the isotope $\hat{A}_2$, also introduced in Santilli (*loc. cit.*)

$$\hat{L}_2: \quad [a,b]_{\hat{A}_2} = a \odot b - b \odot a = aTb - bTa, \tag{5.26}$$

The third, independent isotopy is that attached to $\hat{A}_3$, and it was introduced in Santilli (1981b)

$$\hat{L}_3: \quad [a,b]_{\hat{A}_3} = wawbw - wbwaw, \qquad w^2 = ww \neq 0. \tag{5.27}$$

A fourth isotope is that attached to $\hat{A}_4$, i.e.,

$$\hat{L}_4: \quad [a,b]_{\hat{A}_4} = \text{wawTwbw} - \text{wbwTwaw}, \tag{5.28}$$
$$w^2 = w, \quad w, T \neq 0.$$

A fifth and final (abstract) isotope is that characterized by $\hat{A}_5$, i.e.

$$\hat{L}_5: [a,b]_{\hat{A}_5} = \alpha\, [a,b]_{\hat{A}_4}. \tag{5.29}$$

Again, it is believed that the above five isotopes exhaust all possible abstract Lie algebra isotopies (over a field of characteristics zero), although this property has not been proved to date on rigorous grounds.

Note that the Lie algebra attached to the general Lie-admissible product (5.11) are not conventional, but isotopic. In fact, we can write

$$[a,b]_U = (a,b)_A - (b,a)_A = arb - bsa - bra + asb, \tag{5.30a}$$

$$= aTb - bTa = a{*}b - b{*}a, \tag{5.30b}$$

$$r \neq s, \quad r, s, T \neq 0, \quad T = r + s \neq 0.$$

As a matter of fact, Santilli first encountered the Lie-isotopic algebras by studying precisely the Lie content of the more general Lie-admissible algebras (Santilli (1978a)).

The following property can be easily proved from properties of type (5.30).

LEMMA 5.1: An abstract Lie-Santilli algebra $\hat{L}$ attached to a general, nonassociative, Lie-admissible algebra U, $\hat{L} \approx U^-$, can always be isomorphically rewritten as the algebra attached to an isoassociative algebra $\hat{A}$, $\hat{L} \approx \hat{A}^-$, and vice-versa, i.e.

$$\hat{L} \approx U^- \approx \hat{A}^-. \tag{5.31}$$

The above property has the important consequence that *the construction of the abstract Lie-isotopic theory does not necessarily require a nonassociative enveloping algebra because it can always be done via the use of an isoassociative enveloping algebra.* In turn, this focuses again the importance of knowing all possible isotopes of an associative algebra, e.g., from the viewpoint of the representation theory.

As an example, the studies by Eder ((1981) and (1982)) on a conceivable spin

fluctuation of thermal neutrons caused by sufficiently intense external nuclear fields, are formulated via a flexible *nonassociative* Lie-admissible generalization of the enveloping associative algebra of Pauli's matrices. As such, these studies can be identically reformulated via an *associative-isotopic* enveloping algebra. The consequential simplification of the structure is then expected to permit further physical advances.

Note also that *the construction of the abstract Lie-isotopic theory necessarily requires the isotopies of conventional associative envelopes.*

As typical for all abstract formulations of Lie's theory, the Lie-isotopies indicated above are in a form readily interpretable in terms of operators. As such, they provide the foundations for the operator formulations of the generalized relativities (see, e.g., the isotopic generalization of Wigner's theorem on unitary symmetries in Santilli (1983c)). Note in particular the identification of the inverse of the isounit $\hat{1}$ in the structure of product (5.30).

A primary objective of this monograph is to outline the *classical realizations of the Lie-isotopic product* in such a way to admit a ready identification of the isounit. The latter problem will be the subject of subsequent sections. At this point, we shall review Santilli's classical realizations without the identification of their underlying isounit.

The most general possible, classical realization of Lie-Santilli algebras via functions A(a) and B(a) in $T^*E(r,\delta,\Re)$ with local chart

$$a = (a^{\mu}) = (r, p) = (r^i, p_i), \quad i = 1, 2,..., n, \quad \mu = 1, 2,....., 2n, \tag{5.32}$$

is provided by the *Birkhoffian brackets* (Santilli (1978a), (1982a)), also called generalized Poisson brackets (see, e.g., Sudarshan and Mukunda (1974)),

$$(A , B)_{Birkhoff} = (A,B)_{LI} = \frac{\partial A}{\partial a^{\mu}} \Omega^{\mu\nu}(a) \frac{\partial B}{\partial a^{\nu}}, \tag{5.33}$$

where $\Omega^{\mu\nu}$, called the *Lie-isotopic tensor*, is the contravariant form of (the exact, symplectic, *Birkhoff's tensor* (Santilli (1978a) and (1982a))

$$\Omega = (|\Omega_{\alpha\beta}|^{-1})^{\mu\nu}, \tag{5.34a}$$

$$\Omega_{\mu\nu} = \frac{\partial R_{\nu}(a)}{\partial a^{\mu}} - \frac{\partial R_{\mu}(a)}{\partial a^{\nu}}, \tag{5.34b}$$

where the R's are the so-called *Birkhoff's (1927) functions* , and the symplectic character of the covariant tensor (5.34b) ensures the Lie-Santilli character of brackets (5.33) (see the geometric, algebraic and analytic proofs in Santilli (1982a)).

Recall that, unlike the conventional, abstract, Lie brackets (5.4), the conventional Poisson brackets (5.5) characterize a Lie algebra attached to a *nonassociative* Lie-admissible algebra U, Eq.s (5.12). It is then evident that the covering Birkhoff's brackets (5.33) are also attached to a nonassociative Lie-admissible algebra, although of a more general type (see Santilli (*loc. cit.*) for details).

Numerous other classical Lie-isotopic brackets exist in the literature, the most notable being Dirac's generalized brackets for systems with subsidiary constraints (Dirac (1964)).

Note the lack of identification of the underlying generalized unit in Birkhoff's brackets (5.33), as well as in Dirac's brackets. This problem will be studied in Sect. 9.

Realizations of the abstract isotopes $\hat{U}$ of the Lie-admissible algebras can be easily constructed via the above techniques. For instance, an isotope of the general Lie-admissible product (5.11) is given by

$$\hat{U}: \quad (a \hat{,} b) = wawrwbw - wbwswaw, \qquad w^2 = w, \quad w,r,s \neq 0, \quad r \neq s. \tag{5.35}$$

An isotope of the classical realization (5.12) is then given by

$$\hat{U}: \quad (A \hat{,} B) = \frac{\partial A}{\partial a^{\mu}} S^{\mu\nu}(t,a) \frac{\partial B}{\partial a^{\nu}}, \tag{5.36}$$

where the tensor $S^{\mu\nu}$, called by Santilli (*loc. cit.*) the *Lie-admissible tensor* , is restricted by the conditions of admitting Birkhoff's tensor as the attached antisymmetric tensor, i.e.,

$$S^{\mu\nu} - S^{\nu\mu} \equiv \Omega^{\mu\nu}, \tag{5.37}$$

see Appendix A and, for a detailed study with additional examples, Santilli (1981a).

As recalled in the Introduction, the inception of the notion of algebraic isotopy is rather old, and dated back to the early stages of set theory (Bruck (1958)). Nevertheless, the initiation of the technical studies of the notion can be associated with the development of nonassociative algebras in the middle part of this century

and, more particularly, with the study of Jordan algebras (see the bibliography in nonassociative algebras by Balzer *et al.* (1984)).

As also indicated in Sect. I, the first isotopies of Lie algebras were done by Santilli (1978a, b). Since that time, while the study of more general algebras, such as the Lie-admissible algebras, has received considerable attention in the mathematical literature (see, again, the bibliography by Balzer *et al.* (*loc. cit.*)), no research paper in Lie-isotopic algebras has appeared in the mathematical literature to this writing besides the memoirs Santilli (1991a, b).

I.6: LIE-SANTILLI THEORY

We are now sufficiently equipped to initiate the review of Santilli's characterization of the most general known class of integro-differential systems (1.1), beginning with their algebraic treatment.

The isotopic formulation of enveloping associative algebras, Lie algebras and Lie groups was presented in the original proposal by Santilli (1978a), and are today known under the name of *Lie-Santilli theory* . A first review appeared in the monograph Santilli (1982a). The theory was further developed in Santilli (1988a, b) and (1991a, b). A first independent review was provided by Aringazin *et al.* (1990). In this section we shall outline only those aspects of the theory needed for the remaining sections of this work.

The literature on the conventional formulation of Lie's theory is so wast to discourage even a partial outline. A mathematical treatment of structural theorems on universal enveloping associative algebras and other aspects can be found in Jacobson (1962). A physical treatment of the theory can be found in Gilmore (1974). A classical realization of the theory is available in Sudarshan and Mukunda (1974).

In the following we shall first outline the Lie-Santilli theory in its abstract formulation (i.e., in a formulation admitting a direct interpretation via matrices), and then point out its classical realization (i.e., via functions on the cotangent bundle). To avoid a prohibitive length, our presentation will be mainly conceptual, with no technical developments.

To begin, let us recall that the conventional formulation of Lie's theory is based on the notion of *unit* I realized in its simplest possible form, e.g., via the unit value $1 \in \Re$ for the case of a scalar representation, or the trivial n-dimensional unit matrix $I = \mathrm{Diag.}\ (1,1,...,1)$ for the case of an n-dimensional representation, and so on.

In this case, the *universal enveloping associative algebra* A (Jacobson (1962))

with elements a, b, c, ... over a field F (again assumed of characteristic zero) has the structure

$$A: \quad ab = \text{ass. product}, \qquad Ia = aI = a \quad \text{for all } a \in A. \tag{6.1}$$

The *Lie algebra* L (Jacobson (*loc., cit.*)) is then homomorphic to the antisymmetric algebra A^- attached to A characterized by the familiar commutator

$$L: \quad [a,b]_A = ab - ba \tag{6.2}$$

Connected Lie groups G (Gilmore (1974)) can then be defined via power series expansions in A, according to the familiar form for one dimension

$$G: \quad g(w) = \exp_A{}^{iwx} = I + (iwx) / 1! + (iwx)(iwx) / 2! + ... \tag{6.3}$$
$$w \in F, \qquad x = x^\dagger \in A.$$

with well known generalizations to more than one dimension, as well as to discrete components such as the inversions (Gilmore (*loc. cit.*)).

As recalled in Sect. 1, the central idea of the Lie-Santilli theory is to realize Lie's theory with respect to the most general possible unit $\hat{1}$ which, besides invertibility and Hermiticity, has no restriction on its functional dependence. As such, $\hat{1}$ can have a generally nonlinear dependence on all possible or otherwise needed quantities. For an operator interpretation of the theory (see below for its classical counterpart), such a dependence is on an independent parameter t, coordinates x, velocity $\dot{x} = dx/dt$ or momenta p), accelerations $\ddot{x} = d^2x/dt^2$ (or $\dot{p}$), wavefunctions ψ, their conjugate $\psi^\dagger$, their derivatives $\partial\Psi = \partial\psi/\partial x$ and $\partial\psi^\dagger/\partial x$, etc.,

$$\hat{1} = \hat{1}(t, x, \dot{x}, \ddot{x}, \Psi, \Psi^\dagger, \partial\Psi, \partial\Psi^\dagger, ...) \tag{6.4}$$

Furthermore, Lie's theory is known to be insensitive to the topology of its unit. As a result, the generalized unit $\hat{1}$ can be, not only nonlinear, but also nonlocal in all its variables.

The Lie-Santilli theory therefore has all the necessary characteristics to admit, *ab initio* ,the nonlinear, nonlocal and nonhamiltonian forces of systems (1.1), provided that they are all incorporated in the generalized unit (see the subsequent sections for their analytic representation).

The reader should be aware that representations of nonlocal forces outside the unit of the theory would require a new topology precisely of nonlocal-integral

type which, at any rate, does not appear to be available at the pure mathematical level at this time in a form usable for physical applications.

It is easy to see that the lifting $I \Rightarrow \hat{I}$ requires a necessary, corresponding, generalization of the entire Lie theory. In fact, for $\hat{I}$ to be the left and right unit, the universal enveloping algebra, say ξ, of Lie's theory must be generalized into the form, say, $\hat{\xi}$, which is the same vector space as ξ, but now equipped with the generalized product (5.20), i.e.,

$$\hat{\xi}: \quad a*b = aTb, \tag{6.5}$$

where T is fixed, invertible and Hermitean.

As shown in Sect. 5, the new product a*b is still associative and, for this reason, $\hat{\xi}$ is called *associative-isotopic algebra*, or *isoassociative algebra* for short (Santilli (1978a)). Under the assumption $\hat{I} = T^{-1}$, $\hat{I}$ is indeed the correct right and left unit of the theory, i.e.,

$$\hat{I} = T^{-1}, \tag{6.6a}$$

$$\hat{I}*a = a*\hat{I} \equiv a, \quad \text{for all } a \in \hat{A}, \tag{6.6b}$$

and is called the *isounit* (*loc. cit.*).

Owing to the isotopic character of the generalizations (often referred to as *liftings*), the structural theorems of conventional universal enveloping associative algebras ξ, such as the *Poincaré-Birkhoff [7]-Witt Theorem* for the infinite-dimensional basis (see Jacobson (*loc. cit.*)), admit consistent extensions to the *isoassociative envelope* $\hat{\xi}$, as shown since the original proposal (Santilli (*loc. cit.*)), and it is today called *Poincaré-Birkhoff-Santilli-Witt theorem*.

In particular, the ordered basis $X = (X_i)$, $i = 1, 2, \ldots, n$, of the original Lie algebra L is left unchanged by the isotopy, because Proposition .3.1 applies to L as a vector space, thus preserving the basis X.

In the transition to the underlying associative algebra, we have evidently the same occurrence. However, when ξ is turned into an *isotopic envelope*, the original infinite-dimensional basis of ξ is lifted into the form characterized by the Poincaré-Birkhoff-Santilli-Witt Theorem

$$\hat{\xi}: \ \hat{I}, \ X_i, \ X_i*X_j \ (i \leq j), \ X_i*X_j*X_k \ (i \leq j \leq k), \ldots \tag{6.7}$$

[7] It should be indicated that the name "Birkhoff" here refers to Santilli's former colleague at the Department of Mathematics of Harvard University, G. Birkhoff, son of the author of Birkhoff's equations, G.D.Birkhoff.

For brevity, we refer the interested reader to the reviews by Santilli (1982a) and Aringazin *et al.* (1990). Thus, $\hat{\xi}$ is indeed, a *bona fide, universal enveloping isoassociative algebra*.

Additional associative isotopies independent from form (6.5) are presented in Sect. 5. Isotopy (6.5) is however the fundamental one of this analysis because it admits a right and left generalized unit.

The Lie algebras, say, $\hat{L}$ are now homomorphic to the antisymmetric algebra $\hat{\xi}^-$ attached to $\hat{\xi}$, $\hat{L} \approx \hat{\xi}^-$, with the new product (5.26), i.e.,

$$\hat{L}: \quad [a,b]_{\hat{\xi}} = a*b - b*a = aTb - bTa, \tag{6.8}$$

which verifies the Lie algebra axioms (5.24), while possessing a structure less trivial than the simplest possible Lie product "ab – ba" of current use. For this reason, the algebras $\hat{L}$ have been called *Lie-Santilli algebras*.

The interplay between the algebra $\hat{L}$ and its isoenvelope $\hat{\xi}$ is intriguing. Consider an n-dimensional Lie algebra L with ordered basis X. In the conventional theory, the Poincaré-Birkhoff-Witt Theorem then characterizes the envelope ξ such that

$$\xi = \xi(L), \qquad [\xi(L)]^- \approx L. \tag{6.9}$$

The corresponding context of the covering Lie-Santilli theory is considerably broader. In fact, the isotopic Poincaré-Birkhoff-Witt Theorem now characterizes an isoenvelope $\hat{\xi}$ which, since it is constructed via the original basis of L, was denoted from its original formulation as $\hat{\xi} = \hat{\xi}(L)$ (and not as $\hat{\xi}(\hat{L})$). The novelty is that now, in general, we have

$$[\hat{\xi}(L)]^- \not\approx L, \qquad [\hat{\xi}(L)]^- \approx \hat{L}, \qquad \hat{L} \not\approx L. \tag{6.10}$$

More particularly, the original envelope $\xi(L)$ can characterize only one Lie algebra, the algebra L. On the contrary, Santilli (1982a) has shown that *the infinite number of possible isoenvelopes $\hat{\xi}(L)$ for each given original algebra L can characterize in one, single, unified algorithm $\hat{\xi}(L)$ all possible Lie algebras $\hat{L}$ of the same dimension, with the sole possible exclusion of the exceptional Lie algebras* [8]. It is hoped that, in this, way the reader can begin to see the power of geometric unification of our isotopies.

Again, owing to the isotopic character of the lifting, conventional structural theorems of Lie algebras, such as the celebrated *Lie's First, Second and Third*

[8] This is evidently due to the assumed Hermiticity of the isounit, see later on Sect. 8.

Theorems (Gilmore (1974)), admit consistent isotopic generalizations identified since the original proposal, and are today called *Lie-Santilli Theorems* . We refer the interested reader for brevity to the locally quoted reviews.

Most importantly, the conventional *structure constants* $C_{ij}{}^k$ of a Lie algebra (Gilmore (*loc. cit.*)) are generalized under isotopy into the *structure functions* $\hat{C}_{ij}{}^k(t,x,\dot{x},\ddot{x},...)$ as requested from the Lie-isotopic Second Theorem with isocommutation relations

$$\hat{L}: \quad [X_i,X_j]_{\hat{\xi}} = X_i * X_j - X_j * X_i = \hat{C}_{ij}{}^k(r,x,\dot{x},\ddot{x},...)X_k, \qquad (6.11)$$

where the $\hat{C}$'s are restricted by certain integrability conditions originating from the *Lie-Santilli Third Theorem* (Santilli (*loc. cit.*)).

As we shall see, isotopes $\hat{L}$ with positive-definite isounits of given, conventional, simple Lie algebras L with basis X_i and structure constants $C_{ij}{}^k$, generally admit a reformulation X'_i of the basis (while keeping the isounit unchanged), which recovers the conventional structure constants, i.e., such that

$$\hat{L}: \quad [X'_i,X'_j]_{\hat{\xi}} = X'_i * X'_j - X'_j * X'_i = C_{ij}{}^k X'_k \qquad (6.12)$$

In turn, this evidently proves the local isomorphism of the infinitely possible isotopes $\hat{L}$ with the original simple algebra L, $\hat{L} \approx L$. One should however keep in mind that, for the case of isounits of undefined topology, the isotopes are generally nonisomorphic to the original algebra, $\hat{L} \not\approx L$.

Let us recall that Cartan's classification identifies all nonisomorphic simple Lie algebras of the same dimension, e.g.,

$$\text{Simple 2-dim. algebras: } O(3), \text{ and } O(2.1), \qquad (6.13a)$$

$$\text{Simple 6-dim. algebras: } O(4), \ O(3.1), \text{ and } O(2.2), \qquad (6.13b)$$

etc. (or algebras isomorphic to the above; see, e.g., Gilmore (*loc. cit.*)).

The covering Lie-Santilli theory allows instead the representation of all simple Lie algebras (6.13) of the same dimension with one algorithm: the unique, abstract, simple, Lie-isotopic algebra in n-dimension,

$$\text{Simple 3-dim, isoalgebra } \hat{O}(3), \qquad (6.14a)$$

$$\text{Simple 6-dim. isoalgebra } \hat{O}(6), \tag{6.14b}$$

and similarly for other cases, with the exclusion of the exceptional algebras indicated earlier.

The recovering of different, generally nonisomorphic (e.g., compact or noncompact) algebras is then reduced to the mere realization of the isounit $\hat{1}$. For details on the above "isotopic unification" of simple Lie algebras, see Sect. 8, Figures 8.1 and 8.2, in particular.

A technical knowledge of the above unification is a necessary pre-requisite for understanding certain physical results, such as the geometric unification of Einstein's special relativity in a Minkowski space, with Einstein's gravitation in a Riemannian space, as well as all their isotopic generalizations for the interior problem, which is achieved via one unique, abstract notion, that of the Poincaré-isotopic symmetry, admitting of an infinite number of different realizations, whether in Minkowskian, or in Riemannian or in more general spaces Santilli (1988b, c) and (1991a, b)).

The reader should keep in mind that, in physical applications, the generators have a direct physical meaning. The isotopic algebras with a direct physical meaning therefore remain structures (6.9), while reformulations (6.10) lose the directly physical meaning of the generators and, as such, they generally carry a sole mathematical meaning.

We also recall the *differential rule* for the isocommutators

$$[A*B,C]_{\xi} = A*[B,C]_{\xi} + [A,C]_{\xi}*B, \tag{6.15a}$$

$$[A,B*C]_{\xi} = [A,B]_{\xi}*C + B*[A,C]_{\xi}, \tag{6.15b}$$

which is based on the fact that the *conventional* product AB of elements A and B of the isoassociative envelope has no mathematical or physical meaning in ξ, and must be replaced with the isotopic product A*B.

In particular, this implies that all conventional operations based on multiplications are now inapplicable to the isotopic theory. As an example, the insistence in the use of the *conventional square*

$$a^2 = aa, \tag{6.16}$$

such as the magnitude of the angular momentum

$$J^2 = \sum_{k=1,2,3} J_k J_k, \tag{6.17}$$

within the context of the isoenvelope $\hat{\xi}$ would imply the violation of isolinearity and numerous other inconsistencies (Sect.s 3, 4). The correct quantity is evidently given by the *isotopic square*

$$a^{\hat{2}} = a * a. \tag{6.18}$$

such as the isotopic magnitude of the angular momentum

$$J^{\hat{2}} = \sum_{k=1,2,3} J_k * J_k, \tag{6.19}$$

The lack of knowledge of these basic elements is reason for considerable confusions. In fact, readers not familiar with the Lie-Santilli theory tend to preserve under isotopy the old notion of square, say, of the angular momentum, Eq. (6.17), by therefore resulting in a host of inconsistencies of which they are generally unaware.

Additional isotopies of Lie algebras independent of (6.7) are presented in Sect. 5, although structure (6.7) is the fundamental one for the construction of Santilli's new generation of isotopic classical relativities, as well as for their operator extensions.

Finally, connected Lie groups cannot be any longer defined via power series in ξ (which would violate the linearity condition), and must be defined in the new envelope $\hat{\xi}$ via infinite basis (6.7) with expressions of the following type for one dimension

$$\hat{G}: \ \hat{g}(w) = \exp_{\hat{\xi}}^{iwX} = \hat{1} + (iwX) / 1! + (iwX)*(iwX) / 2! + \ldots$$

$$= \hat{1}\,[\exp_{\xi}^{iwTX}] = [\exp_{\xi}^{iXTw}]\,\hat{1}, \tag{6.20}$$

with corresponding expressions for more than one dimension as well as for discrete components.

The elements $\hat{g}(w)$ cannot evidently verify the old group laws (Gilmore (*loc.cit.*)) but must verify instead the *isotopic group laws*

$$\hat{g}(w) * \hat{g}(w') = \hat{g}(w') * \hat{g}(w) = \hat{g}(w + w'), \tag{6.21a}$$

$$\hat{g}(w) * \hat{g}(-w) = \hat{g}(0) = \hat{1} \tag{6.21b}$$

where the associativity of the isotopic group product $\hat{g}(w) * \hat{g}(w')$ is understood.

All Lie groups verifying the above laws were called *Lie-isotopic groups* (Santilli (1978a)), and are now known as *Lie-Santilli groups*. Again, we refer the interested reader to the locally quoted literature for the isotopic lifting of conventional theorems on Lie groups.

As an example the *isotopic lifting of the Baker-Campbell-Hausdorff Theorem* (Santilli (*loc. cit.*)) is given by

$$[e_{\xi}^{X_i}] * [e_{\xi}^{X_j}] = e_{\xi}^{X_k}, \tag{6.22a}$$

$$X_k = X_i + X_j + [X_i, X_j]_{\xi} / 2 + [(X_i - X_j), [X_i, X_j]_{\xi}]_{\xi} / 12 + \ldots\ldots \tag{6. 22b}$$

and it is today known as the *Baker-Campbell-Hausdorff-Santilli Theorem*.

As now predictable, *the covering Lie-Santilli theory is expected to unify into one, single, abstract, n-dimensional Lie-isotopic group $\hat{G}(n)$ all possible conventional (non-exceptional) Lie groups in the same dimensions $G(n)$.*

With the terms *Lie-Santilli theory* we shall specifically refer to the collection of formulations based on: 1) the universal enveloping isoassociative algebras, 2) the Lie-Santilli algebras, and 3) the Lie-Santilli groups, including all related structural theorems, as well as the remaining compatible aspects, such as the isorepresentation theory.

Needless to say, an inspection of the quoted literature indicates that the theory is just at the beginning and so much remains to be done. Nevertheless, the main structural lines developed so far are sufficient to identify the foundations of Santilli's new relativities.

The Lie-isotopic generalization of the conventional formulation of Lie's theory was submitted along structural lines conceptually similar to those of the *Birkhoffian generalization of Hamiltonian mechanics* (Santilli (1978a) and (1982a)), i.e., under the condition that the generalized theory coincides with the conventional one at the abstract, realization-free level. In fact, *the isoenvelopes $\hat{\xi}$, the Lie-isotopic algebras $\hat{L}$ and the Lie-isotopic groups $\hat{G}$ coincide by construction with the original structures ξ, L and G, respectively, at the abstract, realization-free level.*

Note that, by central assumption, *Santilli's isotopies preserve the generators and parameters of the original group and generalize instead the structure of the group itself in an axiom preserving way.* These features are of central relevance

for the characterization of interior dynamical systems, such as Jupiter when considered as isolated from the rest of the universe, and thus verifying all conventional total conservation laws.

The representation of the time evolution is evidently of fundamental physical importance. In the conventional case it is given by a one-dimensional Lie group G_1 with the Hamiltonian H as generator and time t as parameter. For the Lie-isotopic case, we have instead the more general structure in finite and infinitesimal forms for an arbitrary quantity Q(t)

$$\hat{G}_1 : \quad Q(t) = [e_\xi^{-itH}] * Q(0) * [e_\xi^{itH}] =$$

$$= \hat{1}\,[e_\xi^{-itTH}] * Q(0) * [e_\xi^{iHTt}]\,\hat{1} \qquad = e_\xi^{-itHT}\, Q(0)\, e_\xi^{iTH}, \tag{6.23a}$$

$$i\,\frac{dQ(t)}{dt} = [G,H]_{\hat{A}} = Q*H - H*Q = QTH - HTQ =$$

$$= Q\,T(t,x,p,\dot{p},\psi,\psi^\dagger,\partial\psi,\partial\psi^\dagger\ldots)\,H - H\,T(t,x,\dot{p},\ddot{p},\psi,\psi^\dagger,\partial\psi,\partial\psi^\dagger,\ldots)\,G. \tag{6.23b}$$

characterizing the Lie-isotopic generalization of Heisenberg's equations, called *isoheisenberg's representation,* originally submitted in Santilli (1978b), p. 752.

The corresponding, equivalent, isoschrödinger's representation is then characterized by the right and left modular-isotopic isoeigenvalue equations

$$i\,\partial_t\,|\psi> = H*|\psi> = HT|\psi> = \hat{E}*|\psi> \equiv E|\psi>, \tag{6.24a}$$

$$-i<\psi|\,\partial_t = <\psi|*H = <\psi|TH = <\psi|*\hat{E} \equiv <\psi|E, \tag{6.24b}$$

$$\hat{E} \in \hat{\mathfrak{R}}, \quad E \in \mathfrak{R},\ H \equiv H^\dagger,$$

introduced by Mignani (1982) and Myung and Santilli (1982a).

By inspection, one can see that Eq.s (6.24) represent a physical system with all possible potential forces, characterized by the conventional Hamiltonian as the sum of the kinetic energy and potential energy, $H = T(\dot{x}) + V(t, x, \dot{x})$, with potential

$$F^{SA} = \frac{d}{dt}\frac{\partial V}{\partial \dot{x}} - \frac{\partial V}{\partial x}, \tag{6.25}$$

as well as an additional class of nonlinear, nonlocal and nonhamiltonian forces

beyond the representational capability of the Hamiltonian, characterized precisely by the isotopic element T which, as one can see, *multiplies* the Hamiltonian from the right and from the left.

Eq.s (6.23) and (6.24) are at the foundations of the operator formulation of the Lie–Santilli theory. In fact, as shown in Santilli (1978b) and subsequently developed by Myung and Santilli (1982a), Mignani, Myung and Santilli (1983) and Santilli (1989a, b, c, d), Eq.s (6.23) and (6.24) characterize a generalization of quantum mechanics on a suitable isotopic form of the Hilbert space, called *hadronic mechanics*.

In this case, the generators X are generally expressed via matrices or via local-differential operators, while the isounit $\hat{1}$ is generally represented by a nonlinear and nonlocal, integro-differential operator.

As an example, for the case of two particles with wavepackets $\psi(t,r)$ and $\phi(t,r)$ in conditions of deep mutual penetrations, the isotopic element can be expressed via the form first introduced by Animalu (1991)

$$\hat{1} = e_A^{-itK\int dv\, \psi(t,r)\, \phi(t,r)}, \qquad K \in \Re, \tag{6.26}$$

One can therefore see in this way that for null wave-overlapping, the integral in the exponent of Eq. (6.26) is null, the isounit $\hat{1}$ assumes the conventional trivial value 1, and all Lie-isotopic formulations recover the conventional formulations identically at both the quantum mechanical as well as Lie levels.

In turn, the emerging isotopic generalization of quantum mechanics under isounit (6.26) is useful for a more adequate nonlocal formulation of: bound states of particles at very short distances resulting in nonlocal internal effects, as expected in the structure of hadrons and, to a lesser quantitative extent, in the structure of nuclei (but not in the structure of atoms); a possible nonlocal structure of Cooper pairs in superconductivity; the origin of Bose–Einstein correlations; and all other particle cases where nonlocal conditions are expected to provide experimentally measurable effects.

Realization (6.26) is useful to provide the reader with an illustration of the needed type of nonlocality, as well as of the type of operator Lie–Santilli theory which is expected from the classical formulations of this analysis.

We now pass to the *classical realization of the Lie–Santilli theory*, which is the central topic of the remaining parts of this paper. In this section we shall present only a few introductory notions. The topic is studied in detail at the analytic level in Sect. 7 and at the geometric level in Sect. 9.

Introduce the conventional phase space $T^*E(r,\delta,\Re)$ with local coordinates

$$a = (a^\mu) = (r, p) = (r_i, p_i), \quad i = 1,2,...,n, \quad \mu = 1,2,...,2n, \tag{6.27}$$

where we shall ignore for simplicity of notation any distinction between covariant and contravariant indeces in the r– and p–variables, but keep such a distinction for the a–variables.

As well known, the celebrated Lie's First, Second and Third Theorems provide a direct characterization of the conventional Hamilton's equations without external terms, as presented in their original derivation (Lie (1893)). Therefore, Lie's Theorems provide a direct characterization of the familiar *Poisson brackets* among functions A(a) and B(a) on $T^*E(r,\delta,\Re)$

$$[A, B] = \frac{\partial A}{\partial a^\mu}\omega^{\mu\nu}\frac{\partial B}{\partial a^\nu} = \frac{\partial A}{\partial r_k}\frac{\partial B}{\partial p_k} - \frac{\partial B}{\partial r_k}\frac{\partial A}{\partial p_k}, \tag{6.28}$$

where $\omega^{\mu\nu}$ is the *contravariant, canonical, Lie tensor* with components

$$(\omega^{\mu\nu}) = \begin{pmatrix} 0_{n\times n} & 1_{n\times n} \\ -1_{n\times n} & 0_{n\times n} \end{pmatrix} \tag{6.29}$$

(see the next sections for details and geometrical meaning).

Santilli's primary physical motivation for proposing the Lie–isotopic theory was to show that the isotopic First, Second and Third Theorems characterize a generalization of Hamilton's equations originally discovered by Birkhoff (1927) and called *Birkhoff's equations*, with the ensuing mechanics called *Birkhoffian mechanics*.

In fact, the *Lie–Santilli Theorems* directly characterize the most general possible, regular realization of Lie brackets on $T^*E(r,\delta,\Re)$, given by *Birkhoff's brackets*

$$[A \hat{,} B] = \frac{\partial A(a)}{\partial a^\mu}\,\Omega^{\mu\nu}(a)\,\frac{\partial B(a)}{\partial a^\nu}, \tag{6.30}$$

where $\Omega^{\mu\nu}$, called *contravariant Birkhoff's tensor*, verifies the conditions for brackets (6.30) to be Lie

$$\Omega^{\mu\nu} + \Omega^{\nu\mu} = 0, \tag{6.31a}$$

$$\Omega^{\tau\rho}\frac{\partial\Omega^{\mu\nu}}{\partial a^\rho} + \Omega^{\mu\rho}\frac{\partial\Omega^{\nu\tau}}{\partial a^\rho} + \Omega^{\nu\rho}\frac{\partial\Omega^{\mu\tau}}{\partial a^\rho} = 0. \tag{6.31b}$$

The reader should also be aware that, in a classical realization, the isotopic rules (6.15) no longer hold because the product of functions A and B in the phase space $T^*E(r,\delta,\Re)$ is conventional, AB, and the differential rules for the classical isotopic brackets are given by

$$[A \hat{,} BC] = [A \hat{,} B] C + B [A \hat{,} C], \tag{6.32a}$$

$$[AB \hat{,} C] = [A \hat{,} C] B + A [B \hat{,} C]. \tag{6.32b}$$

A comprehensive treatment of Birkhoffian mechanics was presented in Santilli (1982a). However, such a formulation is local-differential and, thus, basically insufficient for the treatment of systems (1.1) A further structural generalization was therefore needed.

In essence, the abstract formulation of the Lie-Santilli theory, as reviewed in this section, is directly suited for the representation of nonlinear as well as nonlocal interactions, via their embedding in the isounit of the theory.

On the contrary, the classical realization of the Lie-Santilli theory as originally proposed and developed into the Birkhoffian mechanics could indeed represent all possible nonlinear and non-Hamiltonian systems (1.1), but only in their local-differential approximation.

This occurrence is dictated by the use of the conventional *symplectic geometry*, although in its most general possible exact realization precisely given by the Birkhoffian mechanics. The inability to represent any form of nonlocal-integral interactions was then due to the strictly local-differential topology of the underlying geometry.

This created a rather unusual dichotomy whereby the operator formulations of the theory did indeed permit nonlocal interactions (Myung and Santilli (1982a), Mignani, Myung and Santilli (1983)), but their classical counterpart could only admit local interactions (Santilli 1982a)).

This problem was solved only lately via the submission (Santilli (1988a, b), (1991a, b)) of what he called *symplectic-isotopic geometry* and *Birkhoffian-isotopic mechanics* , and now called *Santilli's isosymplectic geometry* and *Birkhoff-Santilli mechanics*, respectively, as the true, classical, geometric and analytic counterparts, respectively, of the abstract Lie-Santilli theory reviewed earlier. We are referring here to the capability of the geometry and of the analytic mechanics to identify in a direct and unambiguous way the underlying isounit in a way similar to the same capability of the Lie-Santilli theory.

Once the applicable geometry and analytic mechanics can directly identify

the underlying isounits, they are readily turned into a form capable of representing integro-differential equations, because one can embed all nonlocal terms in the isounit of the theory.

Santilli's isosymplectic geometry and the Birkhoffian-Santilli mechanics will be reviewed in the next sections. In this section, I shall merely present the main idea of the brackets needed by Santilli.

Let us review the conventional Poisson brackets in the unified notation (6.27) on the conventional 2n-dimensional space $T^*E(r,\delta,\Re)$. Its underlying unit is evidently given by the trivial unit I in 2n-dimension,

$$I = (I_\rho^{\sigma}) = I_{2n\times 2n} = \mathrm{diag}.(1, 1,\ldots, 1). \qquad (6.33)$$

Santilli therefore note that, while the conventional way of writing the Poisson brackets in the disjoint r- and p-coordinates does not allow any effective identification of the underlying unit, this is not the case for the brackets written in the unified a-notation, because they can be written

$$[A\,,B] = \frac{\partial A(a)}{\partial a^\mu}\,\omega^{\mu\rho}\,I_\rho{}^\nu\,\frac{\partial B(a)}{\partial a^\nu}, \qquad (6.34)$$

thus exhibiting the unit of the theory directly in their structure.

The classical isotopic brackets submitted by Santilli (1988a, b), here called *Hamilton-Santilli brackets*, are given by a direct generalization of brackets (6.34) of the form

$$[A\,\hat{,}B] = \frac{\partial A(a)}{\partial a^\mu}\,\omega^{\mu\rho}\,\hat{1}_\rho{}^\nu(t,a,\dot{a},\ldots)\,\frac{\partial B(a)}{\partial a^\nu}, \qquad (6.35)$$

that is, with the following generalization of the canonical tensor

$$\Omega^{\mu\nu} = \omega^{\mu\rho}\,\hat{1}_\rho{}^\nu(t,a,\dot{a},\ldots), \qquad (6.36)$$

evidently under integrability conditions (6.31) for brackets (6.35) to be verify axioms (5.24).

Santilli also writes brackets (6.35) in the disjoint r- and p-coordinates by assuming the diagonal form of the isounit

$$\hat{1} = \mathrm{diag.}\,(\hat{\delta}_{n\times n}, \hat{\delta}_{n\times n}), \quad \hat{\delta} = \hat{\delta}^{\dagger}, \quad \det \hat{\delta} \neq 0, \tag{6.37}$$

and structure (6.29), under which we have

$$[A \hat{,} B] = \frac{\partial A}{\partial r_i}\,\hat{\delta}_{ij}(t,r,p,\ldots)\,\frac{\partial B}{\partial p_j} - \frac{\partial B}{\partial r_i}\,\hat{\delta}^{ij}(t,r,p,\ldots)\,\frac{\partial A}{\partial p_j}, \tag{6.38}$$

One again see in this way that the unified notation (6.27) permits the direct identification of the isounit $\hat{1}$, while such isounit is not directly exhibited by the disjoint r– and p–formulation.

Classical brackets (6.35) do indeed permit the representation of nonlocal systems without any need of introducing a nonlocal topology. This is essentially due to the fact that the canonical structure $\omega^{\mu\nu}$ is preserved in its entirety in structure (6.35), while all nonlocal terms are factorized into the isounit $\hat{1}$, exactly as it occurs at the abstract formulation of the theory.

The only difference between the abstract and classical realizations is that in the abstract case, brackets (6.8) exhibit the presence of the *isotopic element*, while in the classical realization (6.35), the brackets exhibit the presence of the *isounit*.

This is a fully normal occurrence and it is due to the interchange between covariant and contravariant quantities in the transition from the abstract to the classical formulation of the enveloping algebra.

The classical realization of the Lie–isotopic time evolution (6.2a) is straightforward, and it is given by

$$\hat{G}_1:\; Q'(a(t)) = \{\, [e_{\xi}^{\,t\omega^{\mu\alpha}\hat{1}_{\alpha}{}^{\nu}\,(\partial H/\partial a^{\nu})\,(\partial/a^{\mu})}\,]\,\hat{1}\,\}*Q(a(0)), \tag{6.39}$$

which constitutes precisely a classical realization of the abstract Lie–Santilli transformation groups considered earlier, with a ready extension to more than one dimension.

The infinitesimal version of time evolution (6.39) was easily identified by Santilli; it is given by

$$\dot{Q} = [A \hat{,} H] = \frac{\partial Q}{\partial a^{\mu}}\,\omega^{\mu\alpha}\hat{1}_{\alpha}{}^{\nu}\,\frac{\partial H}{\partial a^{\nu}}, \tag{6.40}$$

and characterizes the *Hamilton–Santilli equations* submitted in Santilli (1988a,b), (1991a, d). The more general *Birkhoff–Santilli equations* hold when the factorized Lie tensor is the original Birkhoff's tensor (see next section for details).

The reader should be aware that, while the classical realizations of Lie algebras and groups in their conventional or isotopic realizations are rather simple, as shown above, the classical realizations of universal enveloping associative algebras are rather complex, whether conventional or isotopic, and they will not be treated here for brevity.

In particular, the most notable differences between the abstract isoenvelopes and their classical realizations is the appearance of the so-called *neutral elements* d_{ij} (see, e.g., Sudarshan and Mukunda (1974), p. 222). Conventional closure rules must be generally written

$$L: \qquad [X_i , X_j] = C_{ij}^{\ k} X_k + d_{ij}, \tag{6.41}$$

where the X's are vector-fields on $T^*E(r,\delta,\Re)$, the brackets are the conventional Poisson brackets, and the neutral elements d_{ij} are pure numbers.

In the transition to the Hamilton-Santilli brackets, the situation is predictably more complex, inasmuch as rules (6.41) are now lifted into the isotopic form

$$[X_i \hat{,} X_j] = \hat{C}_{ij}^{\ k}(t,a,\dot{a},..) X_k + \hat{d}_{ij}(t,a,\dot{a},..), \tag{6.42}$$

namely, not only the structure constants $C_{ij}^{\ k}$ are lifted into the structure functions $\hat{C}_{ij}^{\ k}$, but also the constant neutral elements d_{ij} are lifted into the *isoneutral elements* $\hat{d}_{ij}$ with a nontrivial dependence in the local variables.

Now, the elimination of the neutral elements is rather simple at the level of abstract Lie algebras and groups, whether conventional or isotopic. Nevertheless their elimination is rather complex within the context of classical realizations.

This occurrence has direct implications in the identification of the classical realizations of *Casimir invariants* of the Lie-Santilli theory, called *isocasimir invariants,* but not in their abstract (or operator) counterpart. In fact, in the abstract case the isocasimir invariant can be globally identified in a rather simple way, while in the corresponding classical realization, the same isocasimir invariants are generally defined only locally, in the neighborhood of a point of the local variables.

This occurrence can be best illustrated by inspecting the global identification of the isocasimir invariants of the Lorentz-isotopic group in Santilli (1983a), and only the local identification of their classical counterpart in Santilli (1988c), (1991d).

The ultimate roots of this occurrence were identified by Santilli (1978a), and are due to the fact that the envelopes underlying the abstract Lie brackets "ab − ba" or their isotopic generalization "a*b − b*a" are *associative Lie-admissible* in the conventional or isotopic sense,

$$\xi: \; ab = \text{assoc.}, \qquad \xi^{-}: \; ab - ba, \tag{6.43a}$$

$$\hat{\xi}: \; a*b = \text{assoc.}, \qquad \hat{\xi}^{-}: \; a*b - b*a. \tag{6.43b}$$

On the contrary, the envelopes underlying the classical Lie brackets (6.34) or their isotopic generalizations (6.35) are *nonassociative Lie-admissible*, Eq.s (5.12),

$$U: \; \frac{\partial A}{\partial r_k}\frac{\partial B}{\partial p_k} = \text{nonassoc.}, \quad U^{-}: \; \frac{\partial A}{\partial r_k}\frac{\partial B}{\partial p_k} - \frac{\partial B}{\partial r_k}\frac{\partial A}{\partial p_k}, \tag{6.44a}$$

$$\hat{U}: \; \frac{\partial A}{\partial r_i}\hat{\delta}_{ij}\frac{\partial B}{\partial p_j} = \text{nonassoc.}, \; \hat{U}^{-}: \frac{\partial A}{\partial r_i}\hat{\delta}_{ij}\frac{\partial B}{\partial p_j} - \frac{\partial B}{\partial r_i}\hat{\delta}_{ij}\frac{\partial A}{\partial p_j}, \tag{6.44b}$$

We know nowadays how to generalize the Poincaré-Birkhoff-Witt Theorem for isoassociative algebras, but their generalization for nonassociative algebras is known only for *flexible Lie-admissible algebras* (Santilli (1978a), Ktorides, Myung and Santilli (1982a)), namely, for a type of algebra for which no classical realization is known at this writing (Sect. 5).

It is evidently true that the classical Lie algebras and groups can be equivalently formulated via an associative envelope, Lemma 5.1. In fact, the Lie-Santilli expansion (6.39) is precisely of conventionally associative type.

However, such an associative reformulation of nonassociative envelopes implies the appearance of the neutral elements. The difficulties in their elimination at this time therefore rest in our lack of knowledge of the infinite-dimensional basis for the nonassociative envelopes U and Û above.

In conclusion, we do have today a vast mathematical and physical knowledge on the associative envelope of the brackets of quantum mechanics, Heisenberg's brackets AB − BA.

Nevertheless, despite truly voluminous studies initiated by Sophus Lie (1893), and contrary to rather widespread beliefs in mathematical and physical circles, we still do not possess today final mathematical knowledge on the envelope of the corresponding classical brackets, the conventional Poisson brackets (6.28).

I.7: BIRKHOFFIAN MECHANICS

We now pass to the second step in Santilli's characterization of systems (1.1), this time, their analytic representation.

We shall first review the elements of the *Birkhoffian generalization of Hamiltonian mechanics,* or *Birkhoffian mechanics,* as originally derived, that is, via formulations on conventional spaces with the algebra structure being the Lie-Santilli theory, and the underlying geometry being the conventional symplectic geometry.

We shall also review Santilli's *direct universality of Birkhoffian mechanics for local-differential systems* verifying the necessary topological conditions, that is, its capability of representing all possible nonlinear and nonhamiltonian systems of ordinary local-differential equations verifying certain continuity and regularity conditions (universality) directly in the coordinates of their experimental detection (direct universality).

We shall then reformulate the Birkhoffian mechanics in a form, called *Birkhoff-Santilli mechanics,* which is formulated on Santilli's isospaces in such a way to exhibit the isounit of the theory directly in the analytic equations and, therefore, in the Lie-Santilli brackets. The geometric structure of the latter mechanics will be studied in Sec. 9.

The primary reason for such a reformulation was indicated earlier, and it is due to the fact that the Birkhoffian mechanics can only represent local-differential systems because it is based on a geometry, the symplectic geometry, which is strictly local-differential in topological character. The Birkhoff-Santilli mechanics, instead, permits the representation of nonlocal-integral systems under the condition that all the nonlocal terms are incorporated in the isounit of the theory, as permitted by the Lie-isotopic algebra.

In turn, the achievement of a mechanics capable of representing nonlocal interactions is necessary, not only for the classical representation of systems of type (1.1), but also for the operator formulation of the theory. In fact, the interactions of primary interest for the *interior* problem in both classical and particles physics are precisely of nonlocal-integral type.

The studies of this section were initiated by Birkhoff (1927) who identified the central local-differential equations of the new mechanics. However, their algebraic and geometrical structures were unknown. Also, Birkhoff applied his equations to conventional, conservative, Hamiltonian systems, such as the stability of the planetary orbits.

Birkhoff's studies went essentially un-noticed for about fifty one years.

Santilli (1978a) rediscovered the equations, by calling them "Birkhoff's equations", and identified: 1) their algebra structure as being that of the Lie-isotopic theory; 2) their geometric structure as being that of the conventional symplectic geometry in its most general possible exact and local formulation; and 3) the capability of the equations of representing all possible nonlinear and nonhamiltonian systems in local-differential approximation. The name of "Birkhoffian mechanics " was apparently submitted in the quoted memoir for the first time.

A presentation of the foundations of the studies, Helmholtz's (1887) *conditions of variational selfadjointness*, were subsequently presented in the monograph Santilli (1978e), while a comprehensive presentation of Birkhoffian mechanics was provided in the subsequent monograph (1982a).

The *nonrelativistic (relativistic) Birkhoff-Santilli mechanics* were introduced for the first time in Santilli (1988a) (Santilli (1988c)), and then developed in Santilli (1988b) (Santilli (1991d)). In this section we shall present the structure of the mechanics in isospaces of unspecified physical interpretation.

The reader should be aware that the Birkhoff-Santilli mechanics provides the ultimate analytic foundations of the isotopies of Galilei's and Einstein's relativities. No in depth knowledge of the Santilli's relativities can therefore be reached without an in depth knowledge of their analytic structure. The rudimentary outline of this section is basically insufficient for this task, and a study of the original references is essential.

We should finally mention that Santilli presented in the same memoir of (1978a) a still more general mechanics possessing, this time, the broader Lie-admissible and symplectic-admissible structures. This more general mechanics was subsequently studied in detail in the monograph Santilli (1981a). The rudiments of this latter mechanics are presented, for completeness, in the Appendices.

As anticipated in Sect. 4, the primary physical motivation for this latter generalization is the following. Whether conventional or isotopic, Birkhoffian mechanics is an axiom preserving generalization of Hamiltonian mechanics. As such, its primary physical emphasis is in space-time symmetries and related first-integrals which represent total conservation laws. This renders Birkhoffian mechanics ideally suited for the characterization of *closed-isolated* interior systems, such as Jupiter when studied as a whole.

Santilli's more general Birkhoffian-admissible mechanics implies instead a generalization of the axiomatic structure of Hamiltonian mechanics into a form which represents instead the time-rate-of-variations of physical quantities. This renders the Birkhoffian-admissible mechanics particularly suited when studying open-nonconservative interior systems, such as a satellite during penetration in Jupiter's atmosphere considered as external.

Santilli's notations, which are herein adopted, are the following. Manifolds over the reals $\Re$ of arbitrary physical interpretation are indicated with the generic symbol $M(\Re)$. Specific physical interpretation of $M(\Re)$ (such as the Euclidean space over the reals), are generally indicated with different symbols (such as $E(\Re)$).

Generic local coordinates on an N-dimensional manifold $M(\Re)$ are indicated with the symbol x, and their components with the symbol $x = (x^i)$, where for all Latin indeces $i = 1, 2, ..., N$. Coordinates of specific physical interpretation (e.g., the Cartesian coordinates on a Euclidean space) are indicated with generally different symbols (e.g., $r = (r^i)$).

To begin, considered a 2n-dimensional manifold $M(\Re)$ with local coordinates $x = (x^i)$, $i = 1, 2, ..., 2n$, over the reals $\Re$. Let t be an independent variable and $\dot{x} = dx/dt$. Birkhoffian mechanics is based on the most general possible variational principle in $M(\Re)$ which is of linear and first-order character, i.e., depending linearly in the $\dot{x}$'s. Our basic analytic tool is then the *Pfaffian variational principle*

$$\delta\hat{A} = \delta\int_{t_1}^{t_2} (R_i(x)\,\dot{x}^i - B(t,x))\Big|_{\tilde{E}}\, dt = 0, \quad i = 1, 2, ..., 2n, \tag{7.1}$$

here written in its *semiautonomous* form, i.e., with the t-dependence restricted only to the B-function, called by Santilli the *Birkhoffian* because it does not generally represent the *Hamiltonian* $H = T + V$.

When computed along an actual path $\tilde{E}$ of the system, principle (7.1) characterizes the following equations

$$\Omega_{ij}(x)\,\dot{x}^j = \frac{\partial B(t,x)}{\partial x^i}, \quad i, j = 1, 2, ..., 2n, \tag{7.2}$$

called by Santilli the *covariant, semiautonomous Birkhoff's equations*, where

$$\Omega_{ij}(x) = \frac{\partial R_j(x)}{\partial x^i} - \frac{\partial R_i(x)}{\partial x^j}, \tag{7.3}$$

is the *covariant Birkhoff's tensor* hereon restricted to be nowhere degenerate (i.e., $\det(\Omega_{ij}) \neq 0$ everywhere in the region considered).

Santilli then called the *contravariant, semiautonomous Birkhoff's equations* the expressions

$$\dot{x}^i = \Omega^{ij}(x)\,\frac{\partial B(t,x)}{\partial x^j}, \tag{7.4}$$

where

$$\Omega^{ij}(x) = (\,|\,\Omega_{rs}(x)|^{-1}\,)^{ij}, \tag{7.5}$$

is the *contravariant Birkhoff's tensor.*

It is easy to see that the brackets of the algebraic structure of the theory among functions A(x), B(x), ... on $M(\Re)$ are characterized by the contravariant tensor Ω^{ij}

$$[A \hat{,} B] = \frac{\partial A(x)}{\partial x^i}\,\Omega^{ij}(x)\,\frac{\partial B(x)}{\partial x^j}, \tag{7.6}$$

while the covariant tensor Ω_{ij} characterizes the two-form

$$\Omega = d\,\Theta = d\,(R_i dx^i) =$$

$$= \tfrac{1}{2}\left(\frac{\partial R_j}{\partial x^i} - \frac{\partial R_i}{\partial x^j}\right) dx^i \wedge dx^j = \tfrac{1}{2}\Omega_{ij}\, dx^i \wedge dx^j. \tag{7.7}$$

As we shall review in Section 9, two-form (7.7) is the most general possible exact symplectic two-form in local coordinates. This provides the necessary and sufficient conditions for brackets (7.6) to be the most general possible classical, regular , unconstrained[9] Lie-Santilli product on $M(\Re)$ (see the proof in Santilli (1982a)).

Brackets (7.6) were called by Santilli *Birkhoff's brackets* , and this terminology will be kept in this volume. The same brackets are also called *generalized Poisson brackets* in other studies (e.g., Sudarshan and Mukunda (1974)). The *fundamental Birkhoff's brackets* are then given by

$$[x^i \hat{,} x^j] = \Omega^{ij}(x), \tag{7.8}$$

and they play an important role for the classical and operator realization of Lie-Santilli, space-time symmetries.

Other fundamental equations are given by the expression called by Santilli the *Birkhoffian Hamilton-Jacobi equations*

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[9] We should mention for completeness that, in addition to the brackets of this section, we also have those defined on an hypersurface of constraints, as it is the case for Dirac's brackets (Dirac (1964)).

$$\frac{\partial \hat{A}}{\partial t} + B(t,x) = 0, \tag{7.9a}$$

$$\frac{\partial \hat{A}}{\partial x^i} = R_i, \tag{7.9b}$$

which are directly derivable from variational principle (7.1) (under the condition of nowhere degeneracy of Birkhoff's tensors) and, as such, are equivalent to Birkhoff's equations (see Santilli (1982a) for details).

Eq.s (7.9) play a predictable fundamental role for the construction of the operator formulation of the isotopic relativities, although in a reduced isotopic form discussed below.

In Hamiltonian mechanics, one usually assigns the Hamiltonian and then computes the equations of motion, when needed. In Santilli's studies of Birkhoffian mechanics the situation is the opposite. In fact, one starts with an arbitrary nonlinear and nonhamiltonian system and then computes its Birkhoffian representation.

A main result can be formulated as follows.

THEOREM II.7.1 (Direct Universality of Birkhoffian Mechanics for Local First-Order Systems; Santilli (1982a), Theorem 4.5.1, p. 54): All local, analytic, regular, nonautonomous, finite-dimensional, first-order, ordinary differential equations on a 2n-dimensional manifold $M(\Re)$ with local coordinates $x = (x^i)$, $i = 1, 2, ..., 2n$, and derivatives $\dot{x} = dx/dt$ with respect to an independent variable t,

$$\dot{x}^i = \Gamma^i(t, x), \tag{5.10}$$

always admit, in a star-shaped neighborhood of a regular point of their variables, a representation in terms of Birkhoff's equations directly in the local variables at hand, i.e.,

$$\left[\frac{\partial R_j(t, x)}{\partial x^i} - \frac{\partial R_i(t, x)}{\partial x^j} \right] \Gamma^j(t, x) = \frac{\partial B(t, x)}{\partial x^i} + \frac{\partial R_i(t, x)}{\partial t}, \tag{7.11}$$

Namely, for each given vector-field $\Gamma(t, x)$ on $M(\Re)$ verifying the topological conditions of the theorem, one can always construct $(n + 1)$ functions $R_i(t, x)$ and

$B(t, x)$ which characterize Birkhoffian representation (7.11).

The reader should be warned that, as the representation emerges from the techniques of Birkhoffian mechanics, it is generally of the *nonautonomous* type (7.11), even when the equations of motion are *autonomous* . Now, representation (7.11) is certainly correct from an analytic viewpoint, i.e., for the use of variational principle (7.1), the Hamilton-Jacobi theory, etc. However, structure (7.11) is not suitable for a generalization of conventional relativities because it violates the condition for the characterization of any algebra, let alone the Lie-isotopic algebras (see Appendix .A for details).

This requires the reduction of nonautonomous representation (7.11) to the semiautonomous form (7.2) (with a consistent Lie-Santilli structure) via the use of the "degrees of freedom" of the theory which are considerably broader than those of the conventional Hamiltonian mechanics.

We limit here to the indication that the so-called *Birkhoffian gauge transformations*

$$R_i(t, x) \Rightarrow R'_i(x) = R_i(t, x) + \frac{\partial G(t, x)}{\partial x^i}, \tag{7.12a}$$

$$B(t,x) \Rightarrow B'(t,x) = B(t,a) - \frac{\partial G(t, x)}{\partial t}, \tag{7.12b}$$

leave unchanged the integrand of principle (7.1) as well as brackets (7.6) and two-form (7.7), within the fixed system of local coordinates of the vector-field. For other degrees of freedom, see the locally quoted references.

Santilli's formulation of the Birkhoffian mechanics is evidently a *covering* of the conventional Hamiltonian mechanics because:

1) The former mechanics is based on methods (the Lie-Santilli theory) structurally more general then those of the latter mechanics (Lie's theory in its simplest possible realization);

2) The former mechanics represents physical systems (local, but arbitrarily nonlinear and nonhamiltonian systems) which are structurally more general than those represented by the latter systems (local potential systems); and

3) The former mechanics admits the latter as a particular case.

To illustrate the latter occurrence, we now introduce a physical realization of the preceding formulation. Let $E(r,\Re)$ be an n-dimensional Euclidean space with

Cartesian coordinates $r = (r_i)$, and let $p = dr/dt = (p_i)$ be their tangent vectors (the ordinary linear momenta). Then, the 2n-dimensional manifold $M(\mathfrak{R})$ can be interpreted as the cotangent bundle (the conventional phase space) $T^*E(r,\mathfrak{R})$ with local coordinates $a = (a^\mu) = (r_i, p_i)$, $\mu = 1, 2, \ldots, 2n$, and $i = 1, 2, \ldots, n$, where, for simplicity of notation, we shall assume all upper and lower Latin indeces on coordinates and momenta to be equivalent, but preserve the distinction between the Greek upper (contravariant) and lower (covariant) indeces on $T^*E(r,\mathfrak{R})$.

It is then easy to see that the particular case of Birkhoffian mechanics characterized by

$$a = (a^\mu) = (r, p) = (r_i, p_j), \tag{7.13a}$$

$$R = R^\circ = (R^\circ_\mu) = (p, 0) = (p_i, 0_i), \tag{7.13b}$$

$$B = B(t, a) = B(t, r, p) = H(t, r, p) = H(t, a), \tag{7.13c}$$

$$\mu = 1, 2, \ldots, 2n, \quad i = 1, 2, \ldots, n,$$

reproduces the conventional Hamiltonian mechanics in its entirety.

In fact, under values (7.13), Pfaffian principle (7.1) re-acquires its canonical form

$$\delta A = \delta \int_{t_1}^{t_2} [p_i \dot{r}^i - H(t, r, p)]_{|E} dt =$$

$$= \delta \int_{t_1}^{t_2} [R^\circ_\mu(a)\, \dot{a}^\mu - H(t, a)]_{|E} dt = 0, \tag{7.14}$$

the covariant tensor (7.3) assumes the canonical-symplectic value on $T^*E(r,\mathfrak{R})$

$$(\omega_{\mu\nu}) = \left(\frac{\partial R^\circ_\nu}{\partial x^\mu} - \frac{\partial R^\circ_\mu}{\partial x^\nu} \right) = \begin{pmatrix} 0_{n\times n} & -1_{n\times n} \\ 1_{n\times n} & 0_{n\times n} \end{pmatrix} \tag{7.15}$$

with canonical-Lie counterpart

$$(\omega^{\mu\nu}) = (\| \omega_{\alpha\beta} \|^{-1}) = \begin{pmatrix} 0_{n\times n} & 1_{n\times n} \\ -1_{n\times n} & 0_{n\times n} \end{pmatrix} \tag{7.16}$$

Birkhoff's equations (7.2) then reduce to the covariant Hamilton's equations

$$\omega_{\mu\nu}\, \dot{a}^{\nu} = \frac{\partial H(t, a)}{\partial a^{\mu}}, \quad \mu = 1, 2, ..., 2n, \tag{7.17}$$

with contravariant form

$$\dot{a}^{\mu} = \omega^{\mu\nu} \frac{\partial H(t, a)}{\partial a^{\nu}}, \tag{7.18}$$

which, when written in the disjoint coordinates $x = (r, p)$, assumes the familiar form

$$\dot{r}_i = \frac{\partial H(t, r, p)}{\partial p_i}, \qquad \dot{p}_i = - \frac{\partial H(t, r, p)}{\partial r_i}, \tag{7.19}$$

Finally, Birkhoffian Hamilton–Jacobi equations (7.9) assume the familiar canonical form

$$\frac{\partial A}{\partial t} + H(t, r, p) = 0, \tag{7.20a}$$

$$\frac{\partial A}{\partial r_i} = p_i, \qquad \frac{\partial A}{\partial p_i} = 0. \tag{7.20b}$$

Note that, while the canonical action A is independent from the variables p as expressed in Eq.s (7.20b), the Pfaffian action $\hat{A}$ is generally dependent on all a's and, thus also on the momenta, as expressed by Eq.s (7.9b).

This occurrence creates problems in the use of the general equations (7.9) for the construction of an operator image of Birkhoffian mechanics, owing to its excessive generality (e.g., because, after using conventional quantization techniques, it would imply "wavefunctions" ψ depending also on momenta, i.e., $\psi = \psi(t, r, p)$).

Santilli's first motivation for the reformulation of the above mechanics into his isotopic form is therefore of physical character, and consists of the study of Pfaffian variational principles which, while being truly generalized, imply an action independent from the p–variables.

Santilli achieved this objective via the following particular form of the R–functions

$$\hat{R}^{\circ} = (\hat{R}^{\circ}_{\mu}(a)) = (P_i(r, p), 0_i) = (pT_1, 0) = (p_k\, T_i^{\ k}{}_i, 0_i), \tag{7.21}$$

where T_1 is an n×n symmetric, nonsingular and real-values matrix

$$T_1 = (T_{1ij}) = (T_{1\,ji}) = (T_1{}^i{}_j) = (T_{1j}{}^i). \tag{7.22}$$

Realization (7.21) characterizes a phase space $T^*E_1(r,\Re)$ for which principle (7.1) becomes

$$\delta\hat{A}^\circ = \delta\int_{t_1}^{t_2} [\hat{R}^\circ_\mu(a)\,\hat{a}^\mu - H(t,a)]_{|\hat{E}}\, dt$$

$$= \delta\int_{t_1}^{t_2} [p_i T_1{}^{ij}\dot{r}_j - H(t,r,p)]_{|\hat{E}}\, dt = 0, \tag{7.23}$$

and can be interpreted as acting on a 2n-dimensional iso-phase-space $T^*\hat{E}_1(r,\Re)$ equipped with the isounit

$$\hat{1}_1 = \mathrm{diag.}\,(T_1{}^{-1}, T_1{}^{-1}). \tag{7.24}$$

Eq.s (7.9) then become

$$\frac{\partial\hat{A}^\circ}{\partial t} + H(t,r,p) = 0, \tag{7.25a}$$

$$\frac{\partial\hat{A}^\circ}{\partial r_i} = p_j T_1{}^j{}_i, \qquad \frac{\partial\hat{A}^\circ}{\partial p_i} = 0, \tag{7.25b}$$

thus confirming the independence of the generalized action from the velocities, as desired.

Intriguingly, the mapping of Eq.s (.8.25) into an operator form yields exactly isoschrödinger's equations (6.24a), as shown by Animalu and Santilli (1990). We can therefore state today that the Birkhoffian generalization of Hamiltonian mechanics admits as operator image the hadronic generalization of quantum mechanics.

The covariant Birkhoff's tensor characterized by Pfaffian principle (7.23) is given by

$$(\Omega^\circ_{\mu\nu}) = \frac{\partial\hat{R}_\nu}{\partial x^\mu} - \frac{\partial\hat{R}_\mu}{\partial x^\nu} = \begin{pmatrix} 0_{n\times n} & (T_2)_{n\times n} \\ -(T_2)_{n\times n} & 0_{n\times n} \end{pmatrix} =$$

$$= \begin{pmatrix} 0_{n\times n} & (T_{1\,ij} + p_k \frac{\partial T_1{}^k{}_i}{\partial p_j})_{n\times n} \\ -(T_{1\,ij} + p_k \frac{\partial T_1{}^k{}_i}{\partial p_j})_{n\times n} & 0_{n\times n} \end{pmatrix} \tag{7.26}$$

namely, has the factorized structure

$$\hat{\Omega}^\circ = \omega \times \hat{T}_2, \tag{7.27a}$$

$$\hat{T}_2 = \mathrm{diag.}\,(T_2, T_2), \tag{7.27b}$$

$$T_2 = (T_{1\,ij} + p_k \frac{\partial T_1{}^k{}_i}{\partial p_j})_{n\times n}\,, \tag{7.27c}$$

with corresponding two-form

$$\hat{\Omega}^\circ = \hat{\Omega}^\circ_{\mu\nu}\, dx^\mu \wedge dx^\nu = (\omega_{\mu\alpha} \hat{T}_2{}^\alpha{}_\nu)\, da^\mu \wedge da^\nu, \tag{7.28}$$

where the upper script " ° " in structure $\hat{\Omega}^\circ$ stands to indicate that the factorized structure ω is canonical. The covariant analytic equations are given by the *Hamilton-Santilli equations* (Santilli (1988a))

$$\hat{\Omega}^\circ_{\mu\nu}(a)\, \dot{a}^\nu = \omega_{\mu\alpha} \hat{T}_2{}^\alpha{}_\nu(a)\, \dot{a}^\nu = \frac{\partial H(t, a)}{\partial a^\mu}. \tag{7.29}$$

The contravariant Birkhoff's tensor has the structure

$$(\hat{\Omega}^{\mu\nu}) = (\omega^{\mu\nu}) \times \hat{1}_2 = \hat{1}_2 \times (\omega^{\mu\nu}) = (\omega^{\mu\alpha} \hat{1}_{2\alpha}{}^\nu),$$

$$= \begin{pmatrix} 0_{n\times n} & (1_2)_{n\times n} \\ -(1_2)_{n\times n} & 0_{n\times n} \end{pmatrix} \tag{7.30}$$

$$\hat{1}_2 = \hat{T}_2^{-1} = \mathrm{diag.}\,(T_2^{-1}, T_2^{-1}) = \mathrm{diag.}\,(1_2, 1_2), \tag{7.30b}$$

$$1_2 = (T_{1\,ij} + p_k \frac{\partial T_1{}^k{}_i}{\partial p_j})^{-1}, \tag{7.30c}$$

and characterizes the brackets

$$[A \hat{,} B] = \frac{\partial A}{\partial a^{\mu}} \, \omega^{\mu\alpha} \, I_{2\alpha}{}^{\nu}(a,..) \, \frac{\partial B}{\partial a^{\nu}}$$

$$= \frac{\partial A}{\partial r_i} \, I_{2ij}(r,p) \, \frac{\partial B}{\partial p_j} - \frac{\partial B}{\partial r_i} \, I_{2ij}(r,p) \, \frac{\partial A}{\partial p_j}, \tag{7.31}$$

with *contravariant Hamilton-Santilli equations*

$$\dot{a}^{\mu} = \hat{\Omega}^{\mu\nu}(a) \, \frac{\partial H(t, a)}{\partial a^{\nu}} = \omega^{\hat{\mu}\alpha} \, \hat{1}_{2\alpha}{}^{\nu}(a) \, \frac{\partial H(t, a)}{\partial a^{\nu}}, \tag{7.32}$$

which can be written in the disjoint r- and p-coordinates

$$\dot{r}_i = I_{2\,ij}(r, p) \, \frac{\partial H(t, r, p)}{\partial p_j}, \tag{7.33a}$$

$$\dot{p}_i = - I_{2\,ij}(r, p) \, \frac{\partial H(t, r, p)}{\partial r_j}, \tag{7.33b}$$

A few comments are here in order. First, two-form (7.28) remains exact and symplectic, as the reader can verify (see Sect. 9 for details). As a result, brackets (7.31) remain Lie-Santilli under factorization (7.27), provided that the elements I_2 are computed as in Eq.s (7.30c).

Second, we note that *Hamilton-Santilli mechanics characterized by analytic equations (7.30) or (7.32) on $T^*\hat{E}_2(r\hat{\mathfrak{R}})$ does indeed permit the representation of nonlocal-integral interactions, provided that they are all incorporated in the isotopic element $\hat{T}_2$ or, equivalently, in the isounit $\hat{1}_2$*. In fact, the local-differential topology of Hamiltonian mechanics is preserved in its entirety in the factorized canonical forms, while the formulations are insensitive to the possible nonlocality of their units.

Moreover, we note that, when the equations of motion represented by Eq.s (7.30) or (7.32) are written in their second-order form (see Santilli (1982a) for details), they characterize second-order Lagrangians[10].

As a result, analytic equations (7.30) or (7.32) on isospaces $T^*\hat{E}_2(r,\hat{\mathfrak{R}})$ do indeed

[10] see footnote[3] in Sect. 1.

characterize systems of type (1.1), that is, the most general possible class of nonlinear, nonlocal and nonhamiltonian systems (1.1) known at this writing[11]

The reader should finally be aware of the distinction between spaces $T^*\hat{E}_1(r,\hat{\Re})$ and $T^*\hat{E}_2(r,\hat{\Re})$. The former is characterized by a one-form, the integrand of Pfaffian principle (7.23), while the latter is characterized by a two-form, Eq.s (7.28). As a result, they have different isotopic elements, $\hat{T}_1$ and $\hat{T}_2$, and different isounits, $\hat{1}_1$ and $\hat{1}_2$, respectively. The isospace characterizing the Lie-Santilli algebra is evidently that of the analytic equations, $T^*\hat{E}_2(r,\hat{\Re})$.

The extension of the above results to a full isotopy of Birkhoffian mechanics, i.e., for structures (7.28) and (7.30) in which the factorized structures are Birkhoffian, rather than Hamiltonian, is straightforward (see also Sect. 9 for its geometrical treatment).

Santilli (loc,. cit.) reach in this way the following

DEFINITION 7.1: Let $T^*\hat{M}_2$ $(r,\hat{\Re})$ be a 2n-dimensional iso-phase spaces with local coordinates $x = (r, p)$, isofield $\hat{\Re} = \Re\hat{1}_2$ and isounit

$$\hat{1}_2 = = (\hat{1}_{2\alpha}{}^{\beta}) = (\hat{1}_2{}^{\beta}{}_{\alpha}) = (\hat{1}_2{}^{\alpha}{}_{\beta}) = (\hat{1}_{2\beta}{}^{\alpha}) = \mathrm{diag.}\,(I_2, I_2) =$$

$$= (I_{2\,ij}, I_{2\,ij}) = = (I_2{}^{ij}, I_2{}^{ij}) = \hat{T}_2{}^{-1} = (T_2{}^{-1}, T_2{}^{-1}) > 0, \tag{7.34a}$$

$$T_2 = (T_{1\,ij} + p_k \frac{\partial T_1{}^k{}_i(r,p)}{\partial p_j}), \tag{7.34b}$$

with T_I (r,p) being an n×n symmetric, nonsingular and real-value matrix. Then, the "Birkhoff-Santilli equations" are given in their covariant form by

$$\hat{\Omega}_{\mu\nu}(a)\,\dot{a}^{\nu} = [\hat{T}_{2\mu}{}^{\alpha}(a)\,\Omega_{\alpha\nu}(a)]\,\dot{a}^{\nu} = \frac{\partial B(t,a)}{\partial a^{\mu}}, \tag{7.35}$$

with contravariant version

$$\dot{a}^{\mu} = \hat{\Omega}^{\mu\nu}(a)\,\frac{\partial H(t,a)}{\partial a^{\nu}} = [\Omega^{\mu\alpha}(a)\hat{1}_{2\alpha}{}^{\nu}(a)]\,\frac{\partial B(t,a)}{\partial a^{\nu}}, \tag{7.36}$$

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[11] As shown by Jannussis et al. (1983), (1985), the direct universality of the more general Lie-admissible treatment is such to include also finite-difference equations. A similar possibility is expected for the simpler Lie-isotopic treatment, evidently upon embedding of all finite-difference terms in the isounit.

namely, they occur when the contravariant tensor $\hat{\Omega}^{ij}$ (covariant tensor $\hat{\Omega}_{ij}$) is, first, Lie–Santilli (isosymplectic), and, second, admits the factorization of the isounit (isotopic element) of the isospace $T^*\hat{E}_2(r,\hat{\Re})$ in which it is defined

$$(\hat{\Omega}^{\mu\nu}) = (\hat{1}_{2\alpha}{}^{\nu}) \times (\Omega^{\mu\alpha}) \tag{7.37a}$$

$$(\hat{\Omega}_{\mu\nu}) = (\Omega_{\mu\alpha}) \times (\hat{T}_2{}^{\alpha}{}_{\nu}), \tag{7.37b}$$

where Ω^{ij} (Ω_{ij}) are conventional, local–differential contravariant (covariant) Birkhoff's tensors. The "Birkhoff–Santilli mechanics" is the mechanics characterized by the Birkhoff–Santilli equations.

It is easy to see that the Birkhoff–Santilli mechanics is *broader* than the conventional one, trivially, because of the preservation of the most general possible Birkhoff's tensors in its structure, plus the isounit. As a result, the Birkhoffian–isotopic mechanics, not only verifies Theorem 7.1 of Direct Universality, but actually verifies it in an *extended* form inclusive of nonlocal integral terms.

In particular, Birkhoff–Santilli equations (7.32) are expected to be "directly universal" for systems (1.1), although the study of this property was left open by Santilli because not needed for his isotopic relativities.

The Birkhoff–Santilli mechanics is however, excessively broad for our needs. In the subsequent sections of this volume we shall use its particularized form as per the following

DEFINITION 7.2: The "Hamiltonian–Santilli mechanics" is the particular case of the Birkhoff–Santilli mechanics in which the general Birkhoff's tensors are replaced by the canonical ones. The "Hamilton–Santilli equations" on $T^*\hat{E}_2(r,\hat{\Re})$ are therefore given in their covariant form by Eq.s (7.30), and in their contravariant form by Eq.s (7.32), with explicit form (7.33) in the r– and p–coordinates.

The above definitions essentially deal with the so–called *"direct problem"* of analytic dynamics, in which one assigns the Hamiltonian and the isounit and then computes the equations of motion via Eq.s (7.29) or (7.32).

Of particular importance for systems (1.1) is the *"inverse problem"* of analytic dynamics, in which one assigns the equations of motion and must then compute the corresponding isoanalytic representation. This problem was studied in great detail in the monograph (Santilli (1978e)) for the case of local, nonlinear and Hamiltonian systems and in the monograph (Santilli (1982a)) for the case of local, nonlinear and nonhamiltonian systems. Its extension to nonlocal, nonlinear and

nonhamiltonian systems was done in Santilli (1988a), (1991a).

DEFINITION 7.3: Let Γ be a nonlinear, nonlocal and nonhamiltonian vector-field in n-dimension with explicit form of type (1.1). A "direct Hamilton-Santilli representation" exists when one can compute a Hamiltonian H and an isotopic element $\hat{T}_2$ from the given equations of motion, in such a way that the following "isoanalytic representation" holds in the given local coordinates a = (r, p)

$$\hat{\Omega}^{\circ}_{\mu\nu}(a)\,\Gamma^{\nu} = \omega_{\mu\alpha}\,\hat{T}_2{}^{\alpha}{}_{\nu}(a)\,\Gamma^{\nu} = \frac{\partial H(t, a)}{\partial a^{\mu}}, \tag{7.38a}$$

$$\omega_{\mu\nu} = \partial_{\mu}R^{\circ}_{\nu} - \partial_{\nu}R^{\circ}_{\mu}, \quad R^{\circ} = (p, 0). \tag{7.38b}$$

The more general "direct Birkhoff-Santilli representation" of the same vector-field Γ in the same local coordinates a = (r, p) occurs when one can compute a Birkhoffian B, 2n Birkhoffian functions R_{μ} and one isotopic element $\hat{T}_2$ such that the following "isoanalytic representation" holds, again, in the given local variables

$$\hat{\Omega}_{\mu\nu}(a)\,\Gamma^{\nu} = [\hat{T}_{2\mu}{}^{\alpha}(a)\,\Omega_{\alpha\nu}(a)]\,\Gamma^{\nu} = \frac{\partial B(t, a)}{\partial a^{\mu}}, \tag{7.39a}$$

$$\Omega_{\mu\nu} = \partial_{\mu}R_{\nu} - \partial_{\nu}R_{\mu}. \tag{7.39b}$$

The methods for the construction of the above isoanalytic representations for the local subcase have been presented in detail in the monograph (Santilli (1982a), Sect. 4.5 in particular., and they cannot be reviewed here to avoid a prohibitive length. We merely recall that they are based on the construction, first, of an equivalent covariant form of the given vector-field Γ characterized by the multiplication of a regular matrix of *integrating factors* $(C_{\mu\nu})$

$$\dot{a}^{\mu} - \Gamma^{\mu} = 0 \quad \Rightarrow \quad \{C_{\mu\nu}(\dot{a}^{\nu} - \Gamma^{\nu})\}_{|SA} = \{C_{\mu\nu}\dot{a}^{\nu} - D_{\mu}\}_{|SA} = 0,$$

$$\det(C_{\mu\nu}) \neq 0, \qquad D_{\mu} = C_{\mu\nu}\Gamma^{\nu}, \tag{7.40}$$

which verifies all the *conditions of variational selfadjointness* (SA), Theorem 4.1.1, p. .20 of Santilli (1982a), i.e.,

$$C_{\mu\nu} + C_{\nu\mu} \equiv 0, \tag{7.41a}$$

$$\frac{\partial C_{\mu\nu}}{\partial a^{\tau}} + \frac{\partial C_{\nu\tau}}{\partial a^{\mu}} + \frac{\partial C_{\tau\mu}}{\partial a^{\nu}} = 0, \tag{7.41b}$$

$$\frac{\partial C_{\mu\nu}}{\partial t} = \frac{\partial D_{\mu}}{\partial a^{\nu}} - \frac{\partial D_{\nu}}{\partial a^{\mu}}, \tag{7.41c}$$

$$\mu, \nu, \tau = 1, 2, \ldots, 2n$$

which, as one can see, are the covariant version of the algebraic conditions (6.31) (see Santilli (1982a) for details).

Once the selfadjointness of the covariant version of the vector-field is assured, the construction of the R_{μ} and B functions from the equations of motion can be done according to several possible methods of the quoted literature, e.g.,

$$R_{\mu}(t, a) = [\int_0^1 d\tau\, \tau\, C_{\mu\nu}(t, a)\,]\, a^{\nu}, \tag{7.42a}$$

$$B(t, a) = -[\int_0^1 d\tau\, (D_{\mu} + \partial R_{\mu} / \partial t)(t, \tau a)\,]\, a^{\mu}. \tag{7.42b}$$

Note that the conditions of variational selfadjointness essentially imply that the matrix of integrating factors coincides with that of Birkhoff's tensor, i.e.,

$$C_{\mu\nu} = \Omega_{\mu\nu} = \partial_{\mu} R_{\nu} - \partial_{\nu} R_{\mu}. \tag{7.43}$$

The generalization of the above techniques to the case of nonlocal vector-fields Γ is based on the construction of an equivalent covariant form (7.40) of the given vector-field in the same coordinates which verifies the following conditions:

1) The multiplicative matrix of integrating factors $(C_{\mu\nu})$ is decomposable in the form

$$C_{\mu\nu} = \hat{T}_{\mu}{}^{\alpha}\, \Omega_{\alpha\nu}, \tag{7.44}$$

where $\hat{T}_{\mu}{}^{\alpha}$ is totally symmetric and $\Omega_{\alpha\nu}$ is totally antisymmetric

$$\hat{T}_{\mu}{}^{\alpha} \equiv \hat{T}^{\mu}{}_{\alpha}, \qquad \Omega_{\alpha\nu} = -\Omega_{\nu\alpha}; \tag{7.45}$$

2) All nonlocal terms are embedded in the symmetric tensor $\hat{T}_{\mu}{}^{\alpha}$, and the quantity $\Omega_{\alpha\nu}$ admits no nonlocal-integral terms; and

3) the resulting system is variationally selfadjoint

$$\{ \hat{T}_\mu{}^\alpha C_{\alpha\nu} \dot{a}^\nu - D_\mu \}_{|SA} = 0. \tag{7.46}$$

Under these conditions, it is easy to see that the following Birkhoff–Santilli representation holds

$$\hat{T}_\mu{}^\alpha \Omega_{\alpha\nu} \dot{a}^\nu - D_\mu \equiv \hat{T}_\mu{}^\alpha \Omega_{\alpha\nu} \dot{a}^\nu - \frac{\partial B}{\partial a^\mu} = 0. \tag{7.47}$$

where the Birkhoffian functions R_μ and B are computed as in the conventional case.

We now review a few simple example from Santilli (1982a), (1988a), (1991d). The simpler isoanalytic representations (7.38) with underlying variational principle (7.23), Lie-Santilli brackets (7.30) and isosymplectic two-forms (7.28) are sufficient for all practical cases. In fact, the Hamiltonian H can represent the totality of potential forces, as in the conventional theory, while the isounit can represent the totality of nonlocal-integral forces that are admitted by the theory. The broader Birkhoff-Santilli setting is useful for mathematical aspects referred to the direct universality of systems (1.1).

Let us begin with the simplest possible case, the free particle

$$\dot{r} = p/m, \qquad \dot{p} = 0, \tag{7.48}$$

which admits the Hamilton-Santilli representation in isospace

$$\hat{E}(r,\hat{\delta},\hat{\Re}): \ \hat{\delta} = \mathrm{diag.}(b_1{}^2, b_2{}^2, b_3{}^2) = \text{constants} > 0, \tag{7.49a}$$

$$\hat{\Re} = \Re\hat{1}, \qquad \hat{1} = \hat{\delta}^{-1}, \tag{7.49b}$$

in terms of the functions

$$R^\circ = (\hat{p}, 0), \qquad \hat{T} = \mathrm{diag.}(\hat{\delta}, \hat{\delta}), \tag{7.50a}$$

$$B = p^{\hat{2}}/2m, \qquad \hat{p}^{\hat{2}} = p\hat{\delta}p, \tag{7.50b}$$

The reader can therefore see that *the transition from the conventional Hamiltonian mechanics to the covering Hamilton-Santilli mechanics implies the possibility of representing the actual shape of the particle considered* (e.g., an

oblate or prolate spheroidal ellipsoid) via the $\hat{\delta}$ isometric while the equations of motion formally coincide with the conventional ones.

After representing the actual shape of the particle considered, *Santilli's methods can also represent all the infinitely possible deformations of the original shape.* The simplest possible case occurs when the deformation of shape $\hat{\delta} \Rightarrow \hat{\delta}'$ is due to a conventional, external, local-potential force. This is the case when a perfectly spherical charge distribution $\delta = \text{diag.}(1, 1, 1)$ in vacuum is subjected to an external electric field which causes the deformation

$$\delta = \text{diag.}(1, 1, 1) \Rightarrow \hat{\delta}' = \text{diag.}(b_1'^2, b_2'^2, b_3'^2) > 0, \tag{7.51}$$

The equations of motion remain the conventional ones

$$\dot{r}_i = p_i/m, \qquad \dot{p}_i = -(\partial V / \partial r)\, r_i / r, \tag{7.52}$$

while the deformation of shape is represented via the Hamilton-Santilli quantities

$$R^\circ = (p, 0), \qquad \hat{T}' = \text{diag.}(\hat{\delta}', \hat{\delta}'), \tag{7.53a}$$

$$H = p^{\hat{2}} / 2m + V(r), \qquad p^{\hat{2}} = p\hat{\delta}p, \quad r = |\, r\, \hat{\delta}\, r\,|^{\hat{\frac{1}{2}}} \tag{7.53b}$$

where all quantities, including powers and absolute values, are properly written in Santilli's isospace $\hat{E}(r,\hat{\delta},\hat{\Re})$.

In conclusions, *Santilli's isotopic methods apply first to the conventional, local, potential subclass of systems (1.1) by*

1) leaving the equations of motion unchanged;

2) by providing the additional capability of representing the actual shape of the particle considered; as well as

3) of being able to represent all its infinitely possible deformations caused by external forces;

all this at this purely classical and Newtonian level (while preserving the exact rotational symmetry, of course, at the isotopic level, as shown in the next section).

By comparison, the conventional Hamiltonian formulation of contemporary mathematics and physics:

1') can represent the shape of a particle only after the rather complex second

quantization and related form factors;

2') they cannot represent the actual shape of the particle considered (say, an oblate spheroidal ellipsoid) because the form factors are only remnants of the shape itself and, at any rate, they can only admit perfectly spherical objects to avoid the breaking of the conventional rotational symmetry; and

3'); they are structurally unable to represent the deformation of a given original shape, again, to avoid the breaking of the conventional rotational symmetry.

The mathematical and physical advancements for the study of conventional local potential systems in the transition from the conventional Hamiltonian techniques to Santilli's isotopic formulations are then transparent.

Nevertheless, Santilli conceived his isotopic formulations for the representation of the most general known, nonlinear, nonhamiltonian and nonlocal systems (1.1). A typical example is given by the perfectly spherical charge distribution above which performs the transition from motion in vacuum to motion within a physical medium, resulting precisely in contact, nonlinear, nonlocal and nonhamiltonian forces between the sphere and the medium, with evident, consequential deformation of shape.

A simple example occurs when the extended particle experiences a nonlocal-integral force with quadratic damping

$$m\ddot{r} + \gamma\dot{r}^2 \int_\sigma d\sigma\, \mathcal{F}(r) = 0, \tag{7.54}$$

which admits the following Hamilton-Santilli representation

$$R^\circ = (p,0), \tag{7.55a}$$

$$\hat{T} = \mathrm{diag.}\, (\delta' \exp\{ \gamma r \int_\sigma d\sigma \mathcal{F}(r) \},\ \delta' \exp\{ \gamma r \int_\sigma d\sigma \mathcal{F}(r) \}), \tag{7.55b}$$

$$B = p\, (\delta' \exp\{ \gamma r \int_\sigma d\sigma \mathcal{F}(r) \})\, p / 2m. \tag{7.54c}$$

where the dash in the isometric δ' indicates deformation of the original shape δ and the quantities b'_k may depend on suitable parameters such as pressure, density, etc.

An endless variety of examples can then be constructed with infinitely possible combinations of local-potential and nonlocal-nonpotential forces depending on the infinitely possible physical media in which motion occurs.

The reader interested in learning Santilli's isotopic methods is urged to construct a number of isoanalytic representations for given preassigned systems (1.1).

I.8: SANTILLI'S ISOSYMMETRIES

We are now sufficiently equipped to study Santilli's nonlinear and nonlocal symmetries of systems (1.1) on isomanifolds, called *Santilli's isosymmetries* . These generalized symmetries evidently play a fundamental role for his construction of the isotopies of Galilei's, Einstein's special and Einstein's general relativities.

In this section we shall consider two main topics. The first is the notion of isosymmetries as the largest possible nonlinear, nonlocal and noncanonical groups of isometries of given isometric spaces. The second is the notion of isosymmetries of given equations of motion on isomanifolds, with related lifting of Noether's theorem and conservation laws.

The notion of isotopic space-time symmetries was introduced in the original proposal of the Lie-isotopic theory (Santilli (1978a)), although it was formulated in conventional manifolds.

The formulation of space-time symmetries as isosymmetries, that is, as symmetries on isomanifolds, appeared in print, apparently for the first time, in Santilli (1983a) in conjunction with his original construction of the infinite family of isotopes $\hat{O}(3.1)$ of the Lorentz symmetry $O(3.1)$. In fact, the paper first constructed the infinite family of isotopies $\hat{M}$ of the Minkowski space M, then introduced the Fundamental Theorem on Isosymmetries (see below), and finally constructed the isotopies of $\hat{O}(3.1)$.

The operator counterpart of the above results was presented in Santilli (1983c) via an isotopy of Wigner's theorems on unitary symmetries.

The theory was formalized in Santilli (1985a), which constitutes the main reference of this section, and applied to the lifting $\hat{O}(3)$ of the group of rotations in the adjoining paper (Santilli (1985b)).[12]

The second part of this section dealing with the isotopic symmetries of given equations of motions, was first introduced in the monograph Santilli (1982a) as part of the Birkhoffian generalization of Hamiltonian mechanics, including the

[12] We should indicate here that Santilli wrote first, in 1982, the papers (Santilli (1985a and b)) on the background methods and then wrote paper (1983a) on the isotopies of the Lorentz symmetry. The preceding two papers appeared in print some two years after the latter because of extreme editorial oppositions he encountered in their publication which he reported in detail in the paper itself (Santilli (1985a)), p. 26.

isotopic generalization of Noether's theorem, and related conservation laws. As now familiar, the theory was nonlinear and nonhamiltonian but local, owing to the use of conventional local-differential manifolds.

The theory was then resumed in Santilli (1988a, b), (1991a, d) and reformulated as isosymmetries on isomanifolds, including the reformulation of Noether's theorem on an isospace, which constitute the basis of the related review of this section.

To begin, consider a *pseudometric space* M (Sect. 3), here defined as an n-dimensional topological space over the field F of real numbers $\Re$, complex numbers C or quaternions Q with local coordinates $x = (x^i)$, $y = (y^i)$, $i = 1, 2, ..., n$, equipped with a nonsingular, sesquilinear and Hermitean composition (x,y) characterizing the mapping

$$(x,y): \ M \times M \Rightarrow M. \tag{8.1}$$

Let $e = (e^i)$ be the basis of M, and define the *metric tensor* via the familiar form

$$(e_i, e_j) = g_{ij}, \quad g = (g_{ij}). \tag{8.2}$$

The condition of nonsingularity is intended to ensure the existence of the inverse

$$g^{ij} = (\|g_{rs}\|^{-1})^{ij}, \tag{8.3}$$

everywhere in the region considered, which permits the customary raising and lowering of indeces

$$x_i = g_{ij} x^j, \qquad x^i = g^{ij} x_j, \tag{8.4}$$

The condition of sequilinearity

$$((x, \alpha y + \beta z) = \alpha(x,y) + \beta(x,z), \ \ (\alpha x + \beta y, z) = \bar{\alpha}(x,z) + \bar{\beta}(y,z), \tag{8.5}$$

where the upper bar denotes conjugation in the field, permits the realization of the composition in the familiar form

$$(x,y) = x^{\dagger} g \, y = x^i g_{ij} y^j. \tag{8.6}$$

Finally, the condition of Hermiticity implies that

$$x^\dagger (gy) = (g^\dagger x^\dagger) y = (gx^\dagger) y \tag{8.7}$$

by characterizing abstract spaces hereon denote M(x,g,F).

The additional condition of positive-definiteness of the metric g implies that we have a *metric space* as per the definition of Sect. 3. Otherwise we have *pseudometric spaces*.

Let us recall from Sect. 3 that a metric space of particular physical relevance is the three-dimensional *Euclidean space* $E(r,\delta,F)$, with local coordinates $r = (r^i)$ over the fields $F = \Re$ (real), C (complex), Q (quaternion), with composition

$$r^2 = r^i \delta_{ij} r^j \in F, \qquad \delta = \mathrm{diag.}\,(1, 1, 1). \tag{8.8}$$

A pseudometric space also relevant in physic is the (3+1)-dimensional *Minkowski spaces* $M(x,\eta,\Re)$ with local coordinates $x = (r,x^4)$, $x^4 = c_o t$, where c_o represents the speed of light in vacuum, $r \in E(r,\delta,\Re)$, and the composition is given by the familiar expression

$$x^2 = x^\mu \eta_{\mu\nu} x^\nu, \qquad \eta = \mathrm{diag.}\,(1, 1, 1, -1). \tag{8.9}$$

Let us also recall the notion of *isometry* G(m) of a generic manifold M(x,g,F), here defined as *the largest possible m-dimensional Lie group G(m) of linear and local transformations $x \Rightarrow x'$ leaving invariant the composition for the separation $x_1 - x_2$ among two points x_1, x_2 of an n-dimensional manifold $M(x,g,F)$, $F = \Re$, C, Q,*

$$x'_1 - x'_2)^\dagger g\,(x'_1 - x'_2) = (x_1 - x_2)^\dagger g\,(x_1 - x_2), \tag{8.10}$$

(see for details, e.g., Gilmore (1974) and quoted literature).

The connected component $G_o(m)$ of G(m) can be defined as an m-dimensional Lie transformation group on M(x,g,F), i.e., as a topological space $G_o(m)$ equipped with a binary associative composition φ characterizing the mapping

$$\varphi\!: \; G_o(m) \times G_o(m) \Rightarrow G_o(m), \tag{8.11}$$

for G(m) to be a topological Lie group, and the additional mapping

$$f:\ G_o(m) \times M \Rightarrow M, \tag{8.12}$$

characterized by n analytic functions f(w; x) depending on m parameters w and the local coordinates $x \in M$, which verify the conditions for $G_o(m)$ to be a Lie transformation group.

It is finally assumed that $G_o(m)$ is a *linear* transformation group on M(x,g,F), i.e., the f-functions have the particular form

$$x' = f(w; x) = A(w)\, x, \tag{8.13}$$

under which the group conditions can be written

$$A(0) = I, \tag{8.14a}$$

$$A(w)\, A(w') = A(w')\, A(w) = A(w + w), \tag{8.14b}$$

$$A(w)\, A(-w) = I, \tag{8.14c}$$

where I is the trivial identity of Lie's theory and the composition is the associative one.

The isometry $G_o(m)$ can then be defined as the largest possible group of transformations (8.13) leaving invariant separation (8.10), i.e.

$$[(x_1 - x_2)^{\dagger} A^{\dagger}]\, g\, [A(x_1 - x_2)] \equiv (x_1 - x_2)^{\dagger}\, g\, (x_1 - x_2), \tag{8.15}$$

which can hold iff in F

$$A^{\dagger} g\, A = A\, g\, A^{\dagger} = gI, \tag{8.16}$$

and

$$\det A = \pm 1. \tag{8.17}$$

Among the rather large number of methodological aspects needed for a comprehensive characterization of G(m), we now restrict our attention to the following.

I) The *universal enveloping associative algebra* $\xi(G_o(m))$ of the Lie algebra $G_o(m)$ recalled in Sect. 6. For readiness in the comparison of the results under

isotopy, let us recall that the basis of $G_o(m)$

$$X = (X_k), \quad X_k^\dagger = -X_k, \quad k = 1, 2, ..., m, \tag{8.18}$$

must be ordered, and that the envelope $\xi(G_o(m))$ is characterized by the infinite-dimensional basis

$$\xi: \ I, \ X_r, \quad X_r X_s \ \ (r \le s), \quad X_r X_s X_t, \ (r \le s \le t), \ .... \tag{8.19}$$

A generic element of $\xi(G_o(m))$ is then an arbitrary polynomial $P = P(X)$ in the X's. The *center* **C** of $\xi(G_o(m))$ is the set of all elements P which commute with all components X_k of the basis, and can be characterized via the set of all possible scalar multiples of the fundamental unit I in F

$$\mathbf{C} = \{\alpha I \mid \alpha \in F, \ I = \mathrm{diag.}(1, 1, ..., 1)\}, \tag{8.20}$$

where the dimension of I is that of the basis (e.g., for the regular representation of $G_o(m)$, I is the m×m unit, etc.), and I is the right and left unit of Lie's theory

$$I X_r = X_r I = X_r, \ \forall X_r \in \xi. \tag{8.21}$$

II) The *connected Lie group* $G_o(m)$ of transformations on M(x,g,F), which is characterized by exponentiations in $\xi(G_o(m))$ via the infinite basis (8.19). For the case of the right modular transformations (8.13), it can be written in the symbolic form

$$G_o(m): \ A(w) = \prod_{|\xi} e^{X_k w_k}, \tag{8.22}$$

where the exponentiation is the conventional one. Exponentiation (8.22) can then be reduced to the desired form via the *Baker-Campbell-Hausdorff Theorem* (Gilmore (*loc. cit.*)). For the left modular action of $G_o(m)$ on M(x,g,F)

$$x'^\dagger = x^\dagger A^\dagger(w), \tag{8.23}$$

we have the realization

$$G_o(m): \ A^\dagger = (\prod_{|\xi} e^{X_k w_k})^\dagger, \tag{8.24}$$

where the skew-Hermiticity of the basis should be taken into account.

III) The *Lie algebra* $G_o(m)$ of $G_o(m)$, which is homomorphic to the antisymmetric algebra $[\xi(G_o(m))]^-$ attached to the envelope $\xi(G_o(m))$, and it is characterized by the commutation rules

$$G_o(m): \ [X_r, X_s]_\xi = X_r X_s - X_s X_r = C_{rs}{}^t X_t, \tag{8.25}$$

where $X_r X_s$ is the trivial associative product in $\xi(G_o(m))$, and the C's are the structure constants.

Finally, the discrete part D(m) of G(m) is characterized by the inversions

$$D(m): \ x' = Px = -x. \tag{8.26}$$

and constitute an invariant Abelian subgroup of G(m).

As a specific example, the largest possible group of isometries G(m) of a three-dimensional Euclidean space $E(r,\delta,F)$ is the *Euclidean group* (see, e.g., Gilmore (1974) or Sudarshan and Mukunda (1974))

$$E(3) = O(3) \otimes T(3), \tag{8.27}$$

where O(3) is the familiar *group of rotations*, and T(3) is the *group of translations.*

Similarly, the largest possible group of isometries of the (3+1)-dimensional Minkowski space $M(x,\eta,\Re)$ is the *Poincaré group* (*loc. cit.*)

$$P(3.1) = O(3.1) \otimes T(3.1), \tag{8.28}$$

where O(3.1) is the *Lorentz group* and T(3.1) is the *group of translations* in space-time.

We pass now to the study of Santilli's infinitely possible isotopies of each given group of isometry. For this purpose, the first needed notion is that of *isospaces* (Definition 3.2):

a) The infinitely possible isotopes $\hat{M}(x,\hat{g},\hat{F})$ of M(x,g,F), which preserve the dimensionality and local coordinates of M(x,g,F), and generalize instead the metric g and field F into the *isometrics* and *isofields*

$$\hat{g} = Tg, \quad T = T^\dagger, \ \det T \neq 0, \tag{8.29a}$$

$$\hat{F} = F\hat{1}, \quad \hat{1} = T^{-1} \tag{8.29b}$$

respectively, with generic composition

$$(x \hat{,} x) = (x^{\dagger} \hat{g}\, x)\hat{1} = (x^{\dagger}\, Tg\, x)\hat{1} \in \hat{F}. \tag{8.30}$$

2) The infinitely possible isotopes $\hat{E}(r,\hat{\delta},\hat{\mathfrak{R}})$ of the Euclidean space $E(r,\delta,\mathfrak{R})$, called *Santilli's isoeuclidean spaces*, with

$$\hat{\delta} = T_{\delta}\delta \equiv T_{\hat{\delta}}, \qquad \hat{\mathfrak{R}} = \mathfrak{R}\hat{1}_{\hat{\delta}}, \; \hat{1}_{\hat{\delta}} = T_{\delta}^{-1} = \hat{\delta}^{-1}, \tag{8.31}$$

and composition

$$r^{\hat{2}} = (r^{t}\, \hat{\delta}\, r)\hat{1}_{\hat{\delta}} \in \hat{\mathfrak{R}}. \tag{8.32}$$

and

3) The infinitely possible isotopes $\hat{M}(x,\hat{\eta},\hat{\mathfrak{R}})$ of the Minkowski space $M(x,\eta,\mathfrak{R})$, called *Santilli's isominkowski spaces*, with

$$\hat{\eta} = T\eta, \qquad \hat{\mathfrak{R}} = \mathfrak{R}\,\hat{1}_{\hat{\eta}}, \qquad \hat{1}_{\hat{\eta}} = T_{\eta}^{-1}, \tag{8.33}$$

and composition

$$x^{\hat{2}} = (x^{\mu}\, \hat{\eta}_{\mu\nu}\, x^{\nu})\hat{1}_{\hat{\eta}} \in \hat{\mathfrak{R}}, \tag{8.34}$$

We introduce now Santilli's isotransformation theory of Sect. 4 on isospaces $\hat{M}(x,\hat{g},\hat{F})$, i.e., the right, modular-isotopic transformations

$$x' = A * x \overset{\text{def}}{=} ATx. \tag{8.35}$$

where T is the isotopic element of the isospace.

The following important properties then follows.

PROPOSITION 8.1: Given linear and local transformations on a metric or pseudometric space M(x,g,F)

$$x' = A(w)\, x, \quad w \in F, \tag{8.36}$$

their images for the infinitely possible isotopes $\hat{M}(x,\hat{g},\hat{F})$

$$x' = A(w)*x, \tag{8.37}$$

are "isolinear" and "isolocal" in the sense that they are linear and local at the abstract, coordinate-free level, but they are generally nonlinear and nonlocal when projected in the original space $M(x,g,F)$,

$$x' = A*x = A(w)\,T(x, \dot{x}, \ddot{x},...)\,x. \tag{8.38}$$

Santilli (1983), (1985a) studied the groups of *isometries of generic isospaces* $\hat{M}(x,\hat{g},\hat{F})$, namely, the largest possible, m-dimensional groups of isolinear and isolocal transformations, denoted $\hat{G}(m)$, leaving invariant the isoseparation $(x\,\hat{,}\,y)$ on $\hat{F}$.

It is evident that the old group of isometries G(m) cannot act consistently on $\hat{M}(x,\hat{g},\hat{F})$, e.g., because of the violation of the linearity condition and other problems. This renders *necessary* the lifting of Lie's theory, from the conventional formulation outlined earlier in this section, to the Lie-Santilli theory.

We shall therefore assume that $\hat{G}(m)$ admits a connected component $\hat{G}_0(m)$ and a discrete part $\hat{D}(m)$. Suppose that $\hat{G}_o(m)$ is an (abstract) topological space equipped with the isomap

$$\hat{\varphi}:\ \hat{G}_o(m)\,\hat{\times}\,\hat{G}_o(m) \Rightarrow \hat{G}_o(m), \tag{8.39}$$

verifying the conditions for $\hat{G}_o(m)$ to be a Lie-Santilli group (Sect. 6), and equipped with the additional isomap

$$\hat{f}:\ \hat{G}_o(m)\,\hat{\times}\,\hat{M} = \hat{M}, \tag{8.40}$$

characterized by analytic functions $\hat{f}(x,...;\ w)$ depending on the same parameters w and the same local variables x of the original isometry G(m), as well as verifying the Lie-Santilli First, Second and Third Theorems mentioned earlier.

We finally impose that isomap (8.40) is isolinear and isolocal, i.e., of the left and right modular-isotopic type

$$x'^{\dagger} = x^{\dagger}*\hat{A}(w) = x^{\dagger}T\hat{A}(w), \tag{8.41a}$$

$$x' = \hat{A}(w)*x = A(w)Tx. \tag{8.41b}$$

This implies that the elements $\hat{A}(w)$ of $\hat{G}(m)$ verify the *Lie-Santilli group laws*

$$\hat{A}(0) = \hat{1} = T^{-1}, \tag{8.42a}$$

$$\hat{A}(w)*\hat{A}(w') = \hat{A}(w')*\hat{A}(w) = \hat{A}(w+w'), \tag{8.42b}$$

$$\hat{A}(w)*\hat{A}(-w) = \hat{1}, \tag{8.42c}$$

where the product $\hat{A}(w)*\hat{A}(w)$ is isoassociative (Sect. 5), with similar laws for the conjugate elements.

DEFINITION 8.1 (Santilli (1983a)): The group of isometries of an n-dimensional isospace $\hat{M}(x,\hat{g},\hat{F})$, $\hat{F} = \hat{\mathfrak{R}}, \hat{C}, \hat{Q}$, called "Santilli's isometries", is the largest possible, m-dimensional, isolinear and isolocal, Lie-Santilli group $\hat{G}(m)$ of isotransformations (8.41) leaving invariant the isoseparation for the difference $z = x - y$ of two points $x, y \in \hat{M}(x,\hat{g},\hat{F})$,

$$(z' \hat{,} z') = (z*\hat{A} \hat{,} \hat{A}*z) = [(z^{\dagger}*\hat{A}^{\dagger})\, \hat{g}\, (\hat{A}*z)]\,\hat{1} =$$

$$= [[(x^i - x^i)\, T_i{}^j\, A^{\dagger}{}_j{}^r]\, \hat{g}_{rs}\, \{A^s{}_k T^k{}_l (x^l - y^l)]\}\,\hat{1} =$$

$$\equiv (x^r - y^s)\, \hat{g}_{rs}\, (x^s - y^s)\,\hat{1}. \tag{8.43}$$

For the construction of $\hat{G}(m)$ Santilli evidently use his Lie-isotopic theory, with particular reference to:

1': The *universal enveloping isoassociative algebra* $\hat{\xi}(\hat{G}_o(m))$ of $\hat{G}_o(m)$ which, by central assumption, is constructed via the same generators of the original isometry $G_o(m)$, i.e., the ordered basis (8.18). The isotopy $\xi(G_o(m)) \Rightarrow \hat{\xi}(G_o(m))$ is characterized by the Poincaré-Birkhoff-Santilli-Witt Theorem mentioned earlier, with infinite-dimensional isobasis

$$\hat{\xi} : \hat{1}\ ,\ X_k,\quad X_r*X_s\ (r \leq s),\quad X_r*X_s*X_t\ (r \leq s \leq t), \ldots, \tag{8.44}$$

where $\hat{1} = T^{-1}$ is the fundamental isounit of the theory and the product is isoassociative, i.e.,

$$X_r * X_s = X_r T X_s, \tag{8.45a}$$

$$\hat{1} * P = P * \hat{1} = P \quad \forall \quad P \in \hat{\xi}, \tag{8.45b}$$

where P is a generic element of $\hat{\xi}$, i.e., a generic polynomial on the basis X. The *isocenter* $\hat{C}$ of the envelope is then characterized by all elements which isocommute with the basis X and all its possible polynomial forms, and it can be represented via all possible isoscalar multiples of $\hat{1}$ on $\hat{F}$

$$\hat{C} = \{ \hat{\alpha} * \hat{1} \mid \hat{\alpha} \in \hat{F} \}, \tag{8.46}$$

II') The *connected Lie-Santilli group* $\hat{G}_o(m)$, which can be characterized by power series expansions in the new envelope $\hat{\xi}(\hat{G}_o(m))$. For the case of one parameter w and one generator X, these generalized group structures are of type (5.8) and can be written for the m-dimensional case

$$\hat{G}_o(m) = \hat{A}(w) : \prod^*{}_{k=1,\dots,m} e_{|\hat{\xi}}^{X_k w_k} = (\prod_{k=1,\dots,m} e_{|\hat{\xi}}^{X_k * w_k}) \hat{1} =$$

$$\overset{\text{def}}{=} B(w; x, \dots) \hat{1}, \tag{8.47}$$

with composition characterized by the Baker-Campbell-Hausdorff-Santilli Theorem. The conjugate expression is evidently given by

$$\hat{G}_o(m) : \hat{A}(w)^\dagger = \prod^*{}_{k=1,\dots,m} e_{|\hat{\xi}}^{w_k X^\dagger_k} = \hat{1} \, (\prod_{k=1,\dots,m} e_{|\hat{\xi}}^{w_k * X_k} =$$

$$\overset{\text{def}}{=} \hat{1} \, B^\dagger(w; x, \dots). \tag{8.48}$$

III') The *Lie-Santilli algebra* $\hat{\mathbf{G}}_o(m)$ of $\hat{G}_o(m)$ characterized by the Lie-Santilli First, Second and Third Theorems with isocommutation rules

$$\hat{\mathbf{G}}_o(m) : \; [X_r \hat{,} X_s]_{|\hat{\xi}} = [X_r \hat{,} X_s] = X_r * X_s - X_s * X_r$$

$$= \hat{C}_{rs}{}^t(x, \dot{x}, \ddot{x}, \dots) \, X_t, \tag{8.49}$$

where the $\hat{C}$'s are the *structure functions* of $\hat{G}(m)$.

Suppose now that the original group G(m) is an isometry of the original space M(x,g,F), i.e., it verifies conditions (8.15)–(8.17). Santilli then proved that all infinitely possible isotopes $\hat{G}(m)$ of G(m) as constructed above automatically leave invariant the new isocomposition

$$\{ (x-y)^{\dagger} * \hat{A}^{\dagger}\, \hat{g}\, \hat{A} * (x-y) \}\hat{1} \equiv [\, (x-y)^{\dagger}\, \hat{g}\, (x-y) \,]\hat{1}, \tag{8.50}$$

or, equivalently, verifies by construction the property

$$\hat{A}^{\dagger}\hat{g}\ \hat{A} = \hat{A}\,\hat{g}\,\hat{A}^{t} = \hat{g}\hat{1}, \tag{8. 51}$$

with

$$\det.(\hat{A}\hat{g}) = \det B = \pm 1, \tag{8.52}$$

without any need of additional conditions.

In fact, property (8.50) holds for the continuous part in view of the identities

$$e^{-w_k TX_k}\, Tg\, e^{X_k T w_k} \equiv Tg, \tag{8.53}$$

which hold iff the original invariance conditions

$$e^{-w_k X_k}\, g\, e^{X_k w_k}, \tag{8.54}$$

are verified, where the exponentiation in ξ has been omitted for simplicity.

In particular, if the original isometry is the orthogonal group O(n) of an n-dimensional Euclidean space E(r,δ,F), the isometric $\hat{\delta}$ coincides with the isotopic element T (Definition 3.2), and expressions (8.53) reduce to an identity, as one can see in the one-dimensional case

$$e^{-w\hat{\delta}X}\, \hat{\delta}\, e^{X\hat{\delta}w} = \hat{\delta} + w(\hat{\delta}X\hat{\delta} - \hat{\delta}X\hat{\delta}) +$$

$$+ \tfrac{1}{2}w^2(\hat{\delta}X\hat{\delta}X\hat{\delta} - \hat{\delta}X\hat{\delta}X\hat{\delta}) + \dots\dots \equiv \hat{\delta}. \tag{8.55}$$

The isotopies $\hat{D}(m)$ of the discrete component D(m) (8.26) are given by *Santilli's isoinversions*

$$\hat{D}(m): \quad \hat{P}x = P\hat{1}*x = Px = -x, \tag{8.56}$$

where P is the original discrete generator.

The above results can then be expressed as follows.

THEOREM 8.1 (Fundamental Theorem on Isotopic Isometries; Santilli ((1983a) and (1985a)) : Let G(m) be an m-dimensional Lie group of isometries of an n-dimensional metric or pseudometric space M(x,g,F) over the field of real numbers $\Re$, complex numbers C or quaternions Q,

$$G(m): \ x'^{\dagger} = x^{\dagger}A^{\dagger}(w), \quad x' = A(w)\,x, \tag{8.57a}$$

$$[\,(x-y)^{\dagger}\,A^{\dagger}(w)]\,g\,[A(w)\,(x-y)\,] \equiv (x-y)^{\dagger}\,g\,(x-y). \tag{8.57b}$$

$$A^{\dagger}g\,A = A\,g\,A^{\dagger} = gI, \tag{8.57c}$$

$$\det A = \pm 1. \tag{8.57d}$$

Then, Santilli's infinitely possible isotopes $\hat{G}(m)$ of G(m) characterized by the same parameters and generators of G(m), and the infinitely possible, nowhere singular, Hermitean and sufficiently smooth isounits $\hat{1} = T^{-1}$ (isotopic elements T), leave invariant the isocomposition $(x^{\dagger}Tgx)\,\hat{1}$ of the isotopic spaces $\hat{M}(x,\hat{g},\hat{F})$, $\hat{g} = Tg$, $\hat{F} = F\hat{1}$, $\hat{1} = T^{-1}$,

$$\hat{G}(m) = x'^{\dagger} = x^{\dagger}*\hat{A}(w) = x^{\dagger}T\,A^{\dagger}(w), \quad x' = A(w)*x = A(w)Tx, \tag{8.58a}$$

$$[\,(x-y)^{\dagger}*\hat{A}^{\dagger}]\,\hat{g}\,[\hat{A}*(x-y)\,] \equiv (x-y)^{\dagger}\,\hat{g}\,(x-y), \tag{8.58b}$$

$$\hat{A}^{\dagger}\,\hat{g}\,A = \hat{A}\,\hat{g}\,\hat{A}^{\dagger} = \hat{g}\hat{1}, \tag{8.58c}$$

$$\mathrm{Det}\,(\hat{A}\hat{g}) = \det\hat{B} = \pm 1. \tag{8.58d}$$

The following comments are now in order:

1) Each given isometry G(m) admits an infinite number of different isotopes $\hat{G}(m)$ characterized by infinitely possible, different isounits which, from a physical viewpoint, represent the infinitely possible interior physical media.

2) Each of the infinite isotopes can be explicitly computed, from expansions

(8.47), via the sole knowledge of the old isometry $G(m)$ and the isotopic element T.

3) Even though the mathematical formulation can be unified for all isotopes $\hat{G}(m)$, the explicit form of the isotransformations is different for different isounits $\hat{1}$.

4) As indicated earlier, the isotransformations are generally nonlinear, because of the dependence of T.

5) The isotransformations are also generally nonlocal because of the possible integral functional dependence of the isotopic elements T.

6) All isotopes $\hat{G}(m)$ are *coverings* of the original isometry $G(m)$ under the sole condition that the old metric g is admitted as a particular case.

7) All Lie algebras, including that of the isometries $G_o(m)$, admit the following *trivial isotopy* $X_r \Rightarrow \hat{X}_r = X_r\hat{1}$, under which

$$\hat{G}_o(m): \quad [\hat{X}_r, \hat{X}_s]_{|\xi} = \hat{X}_r * \hat{X}_s - \hat{X}_s * \hat{X}_r =$$

$$= \quad [X_r, X_s]_{|\xi}\,\hat{1} = (C_{rs}{}^t\, X_t)\,\hat{1} = C_{rs}{}^t\, \hat{X}_t. \tag{8.59}$$

Santilli *excludes* the above isotopies from Theorem 8.1 because they do not produce the invariance of the new isoseparation, as the reader is encouraged to verify.

8) The dimension m of the original isometries $G(m)$ is preserved by all infinitely possible isotopic isometries $\hat{G}(m)$, as the reader is encouraged to verify. In particular, the condition for closure of $\hat{G}(m)$, Eq.s (8.49) are reducible to those for $G(m)$.

9) The isotopic isometries $\hat{G}(m)$ are generally nonisomorphic to the original symmetry $G(m)$. However, as we shall see in the subsequent chapters, all infinitely possible isotopes $\hat{G}(m)$ can be restricted to be locally isomorphic to the original isometry $G(m)$ under the sole condition of positive– (or negative–) definiteness of the isotopic element T.

To understand the physical relevance of Santilli's Fundamental Theorem 8.1, one should be aware that his isotopic generalizations of Galilei's relativity, of Einstein's special relativity and of Einstein's general relativity (Santilli (1988a, b, c, d)) are particular applications of the theorem.

The first, physically relevant particularization of Theorem 8.1 is given by the following

COROLLARY 8.1.a (Santilli (1985b)): Let O(3) be the simple, three-dimensional orthogonal group of isometries of the three-dimensional Euclidean space $E(r,\delta,\Re)$

over the reals $\Re$,

$$O(3):\quad r'^t = r^t R^t(\theta), \quad \dot{r} = R(\theta)\, r, \tag{8.60a}$$

$$r'^2 = r'^t \delta\, r = r R^t R\, r \equiv r^2 = r^t\, r, \tag{8.60b}$$

$$R^t R = R\, R^t = I, \tag{8.60c}$$

$$\det R = \pm 1. \tag{8.60d}$$

where the θ*'s are the Euler's angles. Then, the infinitely possible isotopic generalizations* $\hat{O}(3)$ *of* $O(3)$, *called "Santilli's isorotational symmetries"* characterized *by the same parameters and generators of* $O(3)$, and a nowhere singular, Hermitean and *sufficiently smooth isounits* $\hat{1} = T^{-1}$ *(isotopic elements* $T\,(r,\ \dot{r},\ \ddot{r},...)$*), leave invariant the corresponding, infinitely possible isocompositions* $(r^t\hat{\delta}\ r)\ \hat{1}$ *of the isoeuclidean spaces* $\hat{E}(r,\hat{\delta},\hat{\Re})$ *with* $\hat{\delta} = T\delta = T,\ \hat{\Re} = \Re\hat{1},\ \hat{1} = T^{-1}$,

$$\hat{O}(m):\ r^{\hat{2}} = (r'^t\ \hat{\delta} r')\,\hat{1} = \{[(r \hat{*} \hat{R}^t(\theta)]\,\hat{\delta}\,[\hat{R}(\theta) \hat{*} r]\}\hat{1} \equiv r^{\hat{2}} = (r^t \hat{\delta}\ r)\,\hat{1}, \tag{8.61a}$$

$$\hat{R}^t \hat{*} \hat{R} = \hat{R} \hat{*} \hat{R}^t = \hat{1} = \hat{\delta}^{-1}, \tag{8.61b}$$

$$\mathrm{Det}\,(\hat{R}\hat{\delta}) = \pm 1. \tag{8.61c}$$

Intriguingly, the isotopes $\hat{O}(r)$ leave invariant all infinitely possible deformations of the sphere, while resulting to be locally isomorphic to $O(3)$ for $T > 0$. In this sense, the isotopes $\hat{O}(3)$ reconstruct as exact the rotational symmetry when broken by ellipsoidical deformations of the sphere (*loc. cit.*). In this way, one can reach a first meaning of the isometric $\hat{\delta}$ as representing the shape of the particle considered.

The fundamental physical application of Santilli's isoeuclidean spaces $\hat{E}(r,\hat{\delta},\hat{\Re})$ is however that of providing the first, rigorous, quantitative, and effective mathematical tool for the study of the nonrelativistic motion of extended particles within inhomogeneous and anisotropic physical media.

In fact, the mutation of the metric $\delta \Rightarrow \hat{\delta} = T\delta$ geometrically represents the underlying homogenous and isotropic space represented by the conventional Euclidean metric δ, and its mutation into an inhomogeneous and anisotropic form caused by the presence of a physical medium represented by $T\delta$.

These mathematical tools have permitted Santilli (1982a), (1988a), (1991d) to

construct an isotopic generalization of Galilei's relativity today called *Santilli's isogalilean relativities* [13], which provide a form-invariant description of the most general dynamical systems known at this writing, systems (1.1), i.e., extended and therefore deformable particles while moving within inhomogeneous and anisotropic material media, thus resulting in the most general known combination of long-range, action-at-a-distance, potential forces, as well as short-range, contact, nonlinear, nonlocal and nonlagrangian-nonhamiltonian forces.

Needless to say. Santilli's isogalilean relativities are a covering of the conventional Galilei's relativity, which is admitted as a particular case for $\hat{1} = 1$, that is, when particles exist physical media and return to free motion in vacuum.

The reader should keep in mind the transition from motion in empty space to motion within a physical medium because it is crucial to understand later on (Sect. 12) the transition from conventional parallel transport and geodesic to Santilli's covering geometrical notions of isoparallel transport and isogeodesics.

Remarkably, *Santilli's isogalilean relativities are directly universal for systems (1.1) and do not need any additional verification, because it is verified by construction for each given system (1.1)*. This is first established by the fact that, while Galilei's symmetry is *imposed* on physical systems, Santilli's isogalilean symmetries are instead *constructed* from each given system (1.1). The generalized relativities are then established by the Theorems of Direct Universality of the Santilli's isotopic methods for systems (1.1) without any need for any further verification.

The structure of Santilli's isogalilean symmetries and further details are provided in the appendices.

A further important case is given by the isotopies of the Lorentz isometry.

COROLLARY 8.1.b (Santilli (1983a), (1988c) and (1991d)): Let $O(3.1)$ be the simple Lorentz group of isometries of the conventional Minkowski space $M(x,\eta,\Re)$ over the reals $\Re$,

$$O(3.1):\quad x^t = x^t\,\Lambda^t(w),\quad x' = \Lambda(w)\,x, \tag{8.62a}$$

$$x'^2 = x^t\Lambda^t\,\eta\,\Lambda\,x \equiv x^t\eta\,x,\quad \eta = \mathrm{diag.}(1,1,1,-1), \tag{8.62b}$$

$$\Lambda^t\,\eta\Lambda = \Lambda\,\eta\,\Lambda^t = \eta I, \tag{8.62c}$$

[13] The plural was suggested by Santilli to stress the fact that, while the conventional Galilei's transformations are unique, there exist infinitely different, covering, isogalilean transformations, evidently because of the infinite number of possible, different, interior physical media.

$$(\det \Lambda) = \pm 1 . \tag{8.62d}$$

where the w's are the conventional six parameters of O(3.1). Then, the infinitely possible isotopes $\hat{O}(3.1)$, called "Santilli's isolorentzian symmetries" characterized by the same parameters and generators of the original group O(3.1), and by nowhere singular, Hermitean and sufficiently smooth isounits $\hat{1}$ (or isotopic elements T(x, ẋ, ẍ,...)), leave invariant the isoseparation $(xT\eta x)\hat{1}$ of the corresponding infinite class of Minkowski-isotopic spaces $\hat{M}(x,\hat{\eta},\hat{\mathfrak{R}})$ with $\hat{\eta} = T\eta$, $\hat{\mathfrak{R}} = \mathfrak{R}\hat{1}$, $\hat{1} = T^{-1}$,

$$\hat{O}(3.1):\ x'^t = x^t{*}\hat{\Lambda}(w),\quad x' = \hat{\Lambda}(w){*}x, \tag{8.63a}$$

$$x^{\hat{2}} = (x^t{*}\hat{\Lambda}^t\,\hat{\eta}\,\hat{\Lambda}{*}x)\,\hat{1} = (x^t\hat{\eta}\,x)\,\hat{1}, \tag{8.63b}$$

$$\hat{\Lambda}^t\hat{\eta}\,\hat{\Lambda} = \hat{\Lambda}\,\hat{\eta}\,\hat{\Lambda}^t = \hat{1}\eta, \tag{8.63c}$$

$$\det.(\hat{\Lambda}\hat{\eta}) = \pm 1. \tag{8.63d}$$

Santilli's isominkowski spaces $\hat{M}(x,\hat{\eta},\hat{\mathfrak{R}})$ provide a geometrization of, this time, the space-time, inhomogeneous and anisotropic character of physical media characterized precisely by the mutation $\eta \Rightarrow \hat{\eta} = T\eta$, thus establishing the first, rigorous and effective mathematical methods for the quantitative study of relativistic dynamics within physical media.

Santilli's (1983a), (1988c), (1991d) isolorentzian symmetries $\hat{O}(3.1)$ characterize a generalization of the special relativity called *Santilli's isospecial relativities* which holds for strictly noneinsteinian conditions, such as relativistic motion of extended and therefore deformable particles within physical media, or the propagation of light within inhomogeneous and anisotropic atmospheres.

Santilli's special relativities are a covering of Einstein's special relativity because:

1) They are based on structurally more general mathematical methods (isotopic generalization of contemporary mathematical structures as outlined in this monograph);

2) They represent structurally more general physical systems (nonlinear, nonlocal and nonlagrangian-nonhamiltonian systems); and

3) They admit Einstein's special relativity as a simple particular case, trivially, for $\hat{1} = I$.

The isospecial relativities result in a series of quantitative predictions of new, measurable, effects, such as the prediction of a redshift of light when propagating within inhomogeneous and anisotropic media and other physical effects structurally outside the technical capabilities of Einstein's special relativity (see the locally quoted literature).

Thus, while Santilli's isogalilean relativities do not need experimental verifications, as indicated earlier, his isospecial relativities require explicit experimental tests.

Nevertheless, to really understand the plausibility of the isospecial relativities, the reader should keep in mind that they admit as particular case the isogalilean relativities via a mere isotopic lifting of the conventional methods of group contraction (see Appendix A of Santilli (1988c), or the monograph (1991d)).

All isotopes $\hat{O}(3.1)$ result to be locally isomorphic to $O(3.1)$ for all isotopic elements $T > 0$, and they constitute the basis for Santilli's isotopies of the special relativity.

Thus, contrary to a rather popular belief in mathematics and physics, a deformation of the Minkowski metric $\eta \Rightarrow \hat{\eta} = T\eta$ DOES NOT imply a necessary breaking of the Lorentz symmetry.

This erroneous belief is due to the restriction of Lie's theory to the simplest possible Lie product "AB − BA". In fact, if one uses the more general product of Santilli's type "ATB − BTA" the validity of the Lorentz symmetry is reestablished in full.

In particular, the isometric $\hat{\eta}$ can be a conventional Riemannian metric $g(x)$ (Corollary 3.2.c). As a result, the Lorentz–isotopic group $\hat{O}(3.1)$ for $T = g$ results to be the global group of isometries of conventional exterior gravitational models, thus creating the possibility of constructing covering gravitational theories for the interior problem via the mere isotopies of isotopies $\eta \Rightarrow \hat{\eta} = T(x)\eta = g(x) \Rightarrow \hat{g} = T'(x, \dot{x}, \ddot{x}, ..)\, g(x)$.

These geometrical results are at the basis of Santilli's interpretation of the general relativity as an isotopy of the special (see Sect.s 11 and 12 for more details).

A further case of physical relevance is the following.

COROLLARY 8.1.c (Mignani (1984)) (Mignani and Santilli (1991): Let SU(3) *be the semisimple special unitary group of isometries of a two-dimensional Euclidean space* $E(x,\eta,C)$ *over the complex field* C

$$SU(3):\ z'^{\dagger} = z^{\dagger}\, U^{\dagger}(w), \quad z' = U(w)\, z, \tag{8.64a}$$

$$z^{\dagger}U^{\dagger}\,\delta\, U\, z = z^{\dagger}\delta\, z, \tag{8.64b}$$

$$U^\dagger U = U U^\dagger = I_{2\times 2}, \tag{8.64c}$$

$$\det. U = +1, \tag{8.64d}$$

Then, the infinitely possible isotopes $\hat{SU}(3)$ of SU(3) characterized by the same parameters and generators of SU(3) and by nowhere degenerate, Hermitean and sufficiently smooth isounits $\hat{1}$ (or isotopic elements $T(z, z^\dagger, ...)$) leave invariant the isotopic separation $(z^\dagger T\delta z)\hat{1}$ of the isotopic spaces $\hat{E}(z,\hat{\delta},\hat{C})$ with $\hat{\delta} = T\delta$, $\hat{C} = C\hat{1}$, $\hat{1} = T^{-1}$,

$$\hat{SU}(3): \quad z'^\dagger = z^\dagger * \hat{U}^\dagger, \qquad z' = \hat{U} * z, \tag{8.65a}$$

$$z'^\dagger \hat{\delta} z' = z^\dagger * \hat{U}^\dagger \hat{\delta} \hat{U} * z = z^\dagger \hat{\delta} z, \tag{8.65b}$$

$$\hat{U}^\dagger \hat{\delta} \hat{U} = \hat{U} \hat{\delta} \hat{U}^t = \hat{1}, \tag{8.65c}$$

$$\det. (\hat{U}\hat{\delta}) = +1. \tag{8.65d}$$

As we shall see in the appendices, the above corollaries is instrumental in introducing Santilli's notion of *"isoparticle"* (or *"isoquarks"*) as an ordinary elementary particle under nonlocal short-range interactions represented by the isounit $\hat{1}$.

By using the Lie-isotopic theory outlined earlier in this section, it is easy to compute explicit examples of Santilli's isorotations and isolorentz transformations, which can be written as follows for an isorotation around the third axis

$$\hat{\delta} = \text{diag.}\,(g_{11}, g_{22}, g_{33}), \tag{8.66a}$$

$$S_{\hat{\delta}}(\theta) = \begin{pmatrix} \cos[\theta_3(g_{11}g_{22})^{\frac{1}{2}}] & g_{22}(g_{11}g_{22})^{\frac{1}{2}}\sin[q_3(g_{11}g_{22})^{\frac{1}{2}}] & 0 \\ -g_{11}(g_{11}g_{22})^{\frac{1}{2}}\sin[\theta_3(g_{11}g_{22})^{\frac{1}{2}}] & \cos[\theta_3(g_{11}g_{22})^{\frac{1}{2}}] & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{8.66b}$$

and for an isolorentz boost (isorotation in the 3–4 plane)

$$x'^1 = x^1,$$

$$x'^2 = x^2, \tag{8.67a}$$

$$x'^3 = \hat{\gamma}(x^3 - \beta x^3),$$

$$x'^4 = \hat{\gamma}(x^4 - \hat{\beta} x^3),$$

$$\beta = v / c_o, \quad \hat{\beta} = v_k b_k / c_o b_4, \qquad \hat{\gamma} = (1 - \hat{\beta}^2)^{\frac{1}{2}}. \tag{8.67b}$$

For the case of an explicit form of the isounitary symmetry $\hat{U}(3)$, we refer the interested reader to Mignani and Santilli (1991) for brevity.

It is an instructive exercise for the interested reader to prove that isotransformations (8.66) and (8.67) do indeed verify the isoinvariance laws (8.61) and (8.63), respectively.

The more general inhomogeneous isotransformations are given in the appendices because it is recommendable for the reader to study first Santilli's isoparallel transport and isogeodesic and then see their isogalilean, isominkowskian and isoriemannian realizations.

In his construction of the generalized relativities, Santilli always imposes the positive-definiteness of the isounit, $\hat{1} > 0$, to ensure the local isomorphisms of the isotopic and conventional symmetries, with consequential covering character of the generalized over the conventional relativities.

However, from a mathematical viewpoint, the restriction $\hat{1} > 0$ can be removed. In this case, *Santilli's isotopic, simple, n-dimensional groups unify in one single algorithm all nonexceptional simple group of Cartan's classification in the same dimension.*

For the case of simple, three-dimensional Lie groups this can be easily seen by allowing metric (8.66a) of isotransformation (9.66b) to possess both positive and negative values.

We reach in this way the conclusion that *the abstract isotope $\hat{O}(3)$ of $O(3)$ with a nowhere singular, Hermitean and diagonal isometric (8.66a)) of unspecified signature provides a single geometric unification of all possible simple, two-dimensional, Lie groups of Cartan's classification* (Santilli (1985b)).

This important property provides another illustration of the rather remarkable possibilities of the Lie-isotopic theory. It can be readily seen from the fact that the isosymmetry $\hat{O}(3)$ in realization (8.66) can interconnects the *compact* realizations $\hat{O}(3) \approx O(3)$ with sig.$\hat{\delta}$ = (+1,+1,+1), to the *noncompact* realizations $\hat{O}(3) \approx O(2.1)$ with sig.$\hat{\delta}$ = (−1,1,+1). The understanding is that Eq.s (8.66) provides the isotopic generalization of O(3) and O(2.1), rather then the conventional transformations themselves. For additional cases, see Figure 8.1.

We now review Santilli (1983a), (1988c) classification and unification of all possible isotopes of the Lorentz group O(3.1). Note that the preceding classification already contains that for the Lorentz group in (2+1)-space-time dimensions. The

extension of the results to the case of (3+1)–space-time dimension is then straightforward, as summarized in Figure 8.2.

$O_o(3)$: $\hat{\delta} = \delta = (+1,+1,+1)$;	$O_o^d(3)$: $\hat{\delta} = -\delta = (-1,-1,-1)$
$O_1(3)$: Sig. $\hat{\delta} = (+1,+1,+1)$;	$O_1^d(3)$: Sig. $\hat{\delta} = (-1,-1,-1)$
$O_2(3)$: Sig. $\hat{\delta} = (+1,+1,-1)$;	$O_2^d(3)$: Sig. $\hat{\delta} = (-1,-1,+1)$
$O_3(3)$: Sig. $\hat{\delta} = (+1,-1,+1)$;	$O_3^d(3)$: Sig. $\hat{\delta} = (-1,+1,-1)$
$O_4(3)$: Sig. $\hat{\delta} = (-1,+1,+1)$;	$O_4^d(3)$: Sig. $\hat{\delta} = (+1,-1,-1)$
$O(3) : \hat{\delta} = \text{diag.}\ (g_{11}, g_{22}, g_{33})$	

FIGURE 8..1: A classification of all possible isotopes $\hat{O}(3)$ of $O(3)$ submitted in Santilli (1985b). They can be presented via the classification of all possible underlying isoeuclidean spaces $\hat{E}(r,\hat{\delta},\hat{\Re})$ or, directly, via the classification of all possible topologies of the isometric $\hat{\delta}$. The first group $O_o(3)$ is the conventional one. The isotopic theory initiates with the isodual $O_o^d(3)$ as per Definition 3.3 which can be formulated only via the use of a bona-fide isounit $\hat{1} = -I$ Then eight classes of isotopes follow, each one with infinite isotopes, grouped into classes connected by isoduality. The last isotope $O(3)$ is the abstract isotope of Eq.s (8.66) unifying all preceding ones.

$O_o(4)$: $\hat{g} = \text{diag.}\ (+1, +1,+1,+1)$	$O_o^d(4)$: $\hat{g} = \text{diag.}\ [-1,-1,-1,-1)$
$O_o(3.1)$: $\hat{g} = \text{diag.}\ (+1,+1,+1,-1)$	$O_o^d(3.1)$: $\hat{g} = \text{diag.}\ (-1,-1,-1,+1)$
$O_o(2.2)$: $\hat{g} = \text{diag.}\ (+1,+1,-1,-1)$	$O_o^d(2.2)$: $\hat{g} = \text{diag.}\ (-1,-1,+1,+1)$
$\hat{O}_1(4)$: sig. $\hat{g} = (+, +, +, +)$	$\hat{O}_1^d(4)$: sig. $\hat{g} = (-, -, -, -)$
$\hat{O}_1(3.1)$: sig. $\hat{g} = (+, +, +, -)$	$\hat{O}_1^d(3.1)$: sig. $\hat{g} = (-, -, -, +)$
$\hat{O}_2(3.1)$: sig. $\hat{g} = (+, +, -, +)$	$\hat{O}_2^d(3.1)$: sig. $\hat{g} = (-, -, +, -)$
$\hat{O}_3(3.1)$: sig. $\hat{g} = (+, -, +, +)$	$\hat{O}_3^d(3.1)$: sig. $\hat{g} = (-, +, -, -)$
$\hat{O}_4(3.1)$: sig. $\hat{g} = (-, +, +, +)$	$\hat{O}_4^d(3.1)$: sig. $\hat{g} = (+, -, -, -)$
$\hat{O}_1(2.2)$: sig. $\hat{g} = (+, +, -, -)$	$\hat{O}_1^d(2.2)$: sig. $g = (-, -, +, +)$
$\hat{O}_2(2.2)$: sig. $\hat{g} = (+, -, +, -)$	$\hat{O}_2^d(2.2)$: sig. $\hat{g} = (-, +, -, +)$
$O(4) : \hat{g} = \text{diag.}\ (g_{11}, g_{22}, g_{33}, g_{44})$	

FIGURE 8.2. The 21 most significant, different isotopes of the Lorentz group in Santilli's classification (1983a), (1988c), (1991a, d) classification of all possible isoorthogonal group in four dimension. The most general possible isotope is the last

one, denoted with $\hat{\mathbf{O}}(4)$ with an arbitrary topology of its isometric, which *unifies all possible six-dimensional simple Lie groups of Cartan's classification.* In particular, this isotope is the abstract Santilli's isolorentzian group of Theorem 5.1. In fact, depending on the local topology of the isometric, $\hat{\mathbf{O}}(4)$ can assume: any one of the six-dimensional simple Lie groups O(4), O(3.1) and O(2.2) (and others locally isomorphic to the latter); any one of their isodual; as well as any one of their isotopes. An infinite number of possible realizations then emerge. They can be first divided into two classes interconnected by isoduality. Then, among each of these classes, only three essential isogroups emerge, those isomorphic to O(4) or O(3.1) or O(2.2). The classification includes the local and global simple symmetries of the special and general relativities for the exterior problem, as well as of their isotopic generalizations for the interior problems, as elaborated in more details in Sect. 12.

The possibilities of geometrical unification offered by the Lie-Santilli theory are therefore remarkable, and expressible via the following

CONJECTURE 8.1: The simple, abstract, n-dimensional isotopes $\hat{G}(n)$ unify in one single algorithm all possible simple, nonexceptional [14] *Lie algebras of the same dimension in Cartan's classification.*

Santilli has proved the above conjecture for the cases n = 2 and 6. Its proof for the general case is left to the interested reader.

We pass now to the study of *Santilli's isosymmetries* of given equations of motion (1.1) on an isospace. Since these equations are represented by the Birkhoff-Santilli equations (Sect. 7), we can effectively restrict our analysis to the isosymmetries of the latter equations. In particular, we shall first review the symmetries of Birkhoff's equations on a conventional manifold, and then generalize them to our isospaces.

Let $E(r,\delta,\Re)$ be the 3N-dimensional Euclidean space of system (1.1) of N particles. Its cotangent bundle $T^*E(r,\delta,\Re)$ is the 6N-dimensional space with the familiar local coordinates $a = (a^\mu) = (r, p) = (r_{ia}, p_{ia})$, $\mu = 1, 2, ..., 6N$, $i = 1, 2, 3$, $a = 1, 2, ..., N$.

The full representation space is then given by the (6N+1)-dimensional space $\Re_t \times T^*E(r,\delta,\Re)$, where $\Re_t$ represents (nonrelativistically) the ordinary time t.

Suppose as a first step that all nonlocal forces in system (1.1) are null (but the vector-field remains nonlinear and nonhamiltonian), and denotes the

[14] We mentioned earlier that the removal of the exceptional Lie algebra is suggested by the assumption of Hermiticity of the isounit. In attempting the proof of Conjecture 8.1 for arbitrary dimensions, we assume the reader is familiar with the unifying power of isofields.

corresponding vector-field with $\Gamma^\circ = (\Gamma^{\circ\mu}(t,a))$. Then, the Theorems of Direct Universality of Birkhoffian mechanics (Sect. 7) ensures that, under the assumed topological conditions, a representation of the vector-field Γ° always exists in terms of Birkhoff's equations in the local coordinates considered, and we shall write

$$\left[\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right]\Gamma^{\circ\nu} = \frac{\partial B}{\partial a^\mu} \pm \frac{\partial R_\mu}{\partial t}, \tag{8.68}$$

A basic notion for the understanding of the isotopic relativities is the behavior of Birkhoff's equations under the most general possible transformations of the local variables.

Recall that Hamilton's equations preserve their form only under a special class of transformations, the canonical ones.

On the contrary, Birkhoff's equations are the most general equations which can be written in $T^*E(r,\delta,\Re)$ with a Lie/symplectic structure. As such, they preserve their form under the most general possible transformations of the local variables.

A detailed treatment of this property is provided in Chapter 5.3 of Santilli (1982a). Here let us illustrate the property by introducing the unified notation

$$b = (b^\mu) = (t,a), \quad \mu = 0, 1, ..., 6N. \tag{8.69}$$

Then Birkhoff's equations (7.2) can be written in the unified form

$$\tilde{\Omega}_{\mu\nu}(b)\, db^\nu = 0, \qquad \mu = 0, 1, ..., 6N, \tag{8.70}$$

where Birkhoff's tensor $\Omega_{\mu\nu}$ in $T^*E(r,\delta,\Re)$ is now extended to the form $\tilde{\Omega}_{\mu\nu}$ in $\Re_t \times T^*E(r,\delta,\Re)$

$$\tilde{\Omega}_{\mu\nu} = \frac{\partial \tilde{R}_\nu(b)}{\partial b^\mu} - \frac{\tilde{R}_\mu(b)}{\partial b^\nu}, \tag{8.71}$$

and $\tilde{R} = (-B, R)$ characterizes the one form in $\Re_t \times T^*E(r,\delta,\Re)$

$$\tilde{\Re}_\mu(b)\, db^\mu = R_\mu(a)\, da^\mu - B(t,a)\, dt, \tag{8.72}$$

namely, it characterizes the complete integrand of variational principle (7.1).

Eq.s (8.70) directly reproduces Eq.s (8.68) for $\mu = 1, 2, ..., 6N$, while the additional equation fo $\mu = 0$ yields the identity

$$\left(\frac{\partial B}{\partial a^{\nu}} + \frac{\partial R_{\nu}}{\partial t} \right) da^{\nu} =$$

$$= \left(\frac{\partial B}{\partial a^{\nu}} + \frac{\partial R_{\nu}}{\partial t} \right) \Omega^{\nu a} \left(\frac{\partial B}{\partial a^{\alpha}} + \frac{\partial R_{\alpha}}{\partial t} \right) \equiv 0. \tag{8.73}$$

What we have done here is performed the transition from the *symplectic geometry* in $T^*E(r,\delta,\Re)$, to the so-called *contact geometry* in $\Re_t \times T^*E(r,\delta,\Re)$ (see, e.g., Abraham and Marsden (1967); and Santilli (1982a) for a specific treatment of Birkhoff's equations in the contact geometry). Equivalently, we can see that Birkhoff's tensor in $T^*E(r,\delta,\Re)$ is of symplectic type while its extended version (8.71) is of contact type.

Once the contact character of tensor (8.71) is understood, one can readily see the invariance of Birkhoff's equations (8.70) under the local, but most general possible, smoothness and regularity preserving transformations in $\Re_t \times T^*E(r,\delta,\Re)$

$$b = (t, a) \Rightarrow b' = b'(b) = (t', a') = (t'(t, a), a'(t, a)), \tag{8.74}$$

In fact, contact tensor (8.71) transforms as follows

$$\tilde{\Omega}_{\mu\nu}(b) \Rightarrow \tilde{\Omega}'_{\mu\nu}(b'(b)) = \frac{\partial b^{\mu}}{\partial b^{\alpha}} \tilde{\Omega}_{\alpha\beta}(b(b')) \frac{\partial b^{\nu}}{\partial b^{\beta}}, \tag{8.75}$$

by evidently preserving its structure. The form invariant (but not the symmetry) of Birkhoff's equations then follows.

The physical implications of the above findings are the following. The space-time symmetries of contemporary relativities for motion in vacuum are, first of all, canonical, and then symmetries of the system considered. In the transition to Santilli's motion within physical media all smoothness and regularity preserving transformations are "canonical" and, therefore possible candidates for interior symmetries.

DEFINITION 8.2 (Santilli (1982a)): The local, but most general possible smoothness and regularity preserving transformations (8.74) on $\Re_t \times T^*E(r,\delta,\Re)$ constitute a

"symmetry of Birkhoff's equations", when they leave invariant Birkhoff's tensor in its contact form, i.e., when Eq.s (8.75) implies the particular form

$$\tilde{\Omega}_{\mu\nu}(b) \Rightarrow \tilde{\Omega}'_{\mu\nu}(b') \equiv \tilde{\Omega}_{\mu\nu}(b') \tag{8.76}$$

or, alternatively, when the underlying contact one-form (8.72) is invariant up to Birkhoffian gauge transformations, i.e.

$$\tilde{R}'_{\mu}(b')\, db'^{\mu} = [\, \tilde{R}_{\mu}(b') + \frac{\partial \tilde{G}(b')}{\partial b'^{\mu}} \,]\, db'^{\mu}, \tag{8.77}$$

We now review the construction of *first integrals* (i.,e., conserved quantities) from a given symmetry of Birkhoff's equations.

THEOREM 8.2. (Birkhoffian Noether's Theorem; Santilli (1982a)): If Birkhoff's equations admit a symmetry under an r-dimensional, connected Lie group G_r of infinitesimal transformations

$$G_r:\; b \Rightarrow b' = b + \delta b = (b^{\mu} + w^i\, a^{\mu}{}_i(b)) = \\ = \begin{pmatrix} t + w^i p_i(t, a), \\ a^{\mu} + w^i\, \eta^{\mu}{}_i(t, a) \end{pmatrix} \tag{8.78}$$

then there exist r first integrals $\mathcal{F}_i$ (b) of the equations of motion which are conserved along an actual path $\tilde{E}$

$$\frac{d}{dt}\mathcal{F}_i(b)|_{\tilde{E}} \equiv 0, \tag{8.79}$$

namely, there exist r linear combinations of Birkhoff's equations which are exact differentials along $\tilde{E}$, i.e.,

$$\frac{d}{dt}\mathcal{F}_i(b) = \tilde{\Omega}_{\mu\nu}(b)\, b^{\nu}\, a^{\mu}{}_i\,, \tag{8.80}$$

given explicitly by

$$\mathcal{F}_i(b) = \tilde{R}_{\mu} a^{\mu}{}_i =$$

$$= R_{\mu}(t, a)\, \eta^{\mu}{}_{i}(t, a) - B(t, a)\, \rho(t, a) + G(_{i}(t, a). \qquad (8.81)$$

Note that the "new time" t′ in Birkhoffian mechanics is a function of the old time t as well as of the coordinates r and momenta p,

$$t' = t(t, r, p). \qquad (8.82)$$

This property is important to understand the isotranslations in time of the isotopic relativities (see the isogeodesics of Sect. 12 and the appendices).

Intriguingly, this property is typical of relativistic formulations but not of Hamiltonian mechanics. Santilli's isogalilean relativities then achieve a form of symmetric behavior of time for both nonrelativistic and relativistic formulations.

Note that the symmetry G_{Γ} of Theorem 8.2 is a *conventional Lie symmetry* defined on a *conventional space*.

Recall also that the nonautonomous Birkhoff's equations considered until now in this section do not admit a consistent algebraic structure (Appendix C). From now on we shall therefore restrict our attention to the semiautonomous case.

We are now in a position to consider the Birkhoff–Santilli representation of systems (1.1) on isospaces $T^*\hat{E}_2(r,\hat{\delta},\hat{\Re})$, i.e.,

$$\hat{T}_{2\mu}{}^{\alpha}(a)\, \Omega_{\mu\alpha}(a)\, \Gamma^{\nu}(t,a) = \frac{\partial B(t, a)}{\partial a^{\mu}}, \qquad (8.83)$$

where $\Omega_{\mu\nu}$ is the conventional, local, Birkhoff–symplectic tensor, and $\hat{T}_2$ the isotopic element of $T^*\hat{E}_2(r,\hat{\delta},\hat{\Re})$.

Our objective here is, not only that of reviewing isosymmetries on isospaces, but also that of reviewing their most general known nonlinear, nonlocal and nonhamiltonian form.

DEFINITION 8.3 (Santilli (1988b), (1991d)): An r–dimensional symmetry of Birkhoff–Santilli equations (8.83) is a "Santilli's isosymmetry" $\hat{G}_r$ when it is defined on isospaces $T^*\hat{E}_2$ $(r,\hat{\delta},\hat{\Re})$ and admits infinitesimal transformations of the Lie-isotopic type

$$a'^{\mu} = a^{\mu} + w^{i}\, \Omega^{\mu\alpha}(a)\, \hat{1}_{2\alpha}{}^{\nu} \frac{\partial X_i}{\partial a^{\nu}}, \qquad (8.84)$$

where $\hat{1}_2 = \hat{T}_2^{-1}$ *is the basic isounit of the isospace, the w's are the parameter and the X's are the generators of* $\hat{G}_r$ *with isocommutation rules*

$$[X_r \hat{,} X_s] = \frac{\partial X_r}{\partial a^\mu} \Omega^{\mu\alpha} \hat{1}_{2\alpha}{}^\nu \frac{\partial X_s}{\partial a^\nu} = \hat{C}_{rs}(a)\, X_s, \tag{8.85}$$

It is easy to see that a *necessary* condition for transformations a $\Rightarrow$ a' to be a symmetry of the Birkhoff-isotopic equations is that they have a Lie-Santilli structure. This renders *necessary* the use of the Lie-Santilli theory for the study of isosymmetries and their first integrals.

THEOREM 8.3 (Integrability Conditions for the Existence of an Isosymmetry; Santilli (1982a) and (1991b)): Necessary and sufficient conditions for a smoothness and regularity preserving transformation (8.84) to be an isosymmetry of the Birkhoff-Santilli equations (8.83) is that they leave the Birkhoffian invariant, i.e.,

$$B'(a') = B(a) + w_i\, [X_i \hat{,} B] \equiv B(a), \tag{8.86}$$

which can hold iff the Birkhoffian B isocommutes with all generators X_I, *i.e.,*

$$[X_i \hat{,} B] \equiv 0, \quad i = 1, 2, ..., r. \tag{8.87}$$

The construction of the isosymmetries of a given system (1.1) is now straightforward. Consider first the conservative part of the given system (1.1), i.e., the vector field

$$\Phi = (\Phi^\mu) = \begin{pmatrix} p_{ia}/m_a \\ -(\partial V / \partial r_{ab})\, \partial r_{ab} / \partial r_{ia} \end{pmatrix} \tag{8.88}$$

which is represented via the familiar Hamilton's equation with Hamiltonian

$$H = T(p) + V(r) = p_{ia}\, \delta_{ij}\, p_{ja}/2m_a + V(r_{ab}), \tag{8.89a}$$

$$r_{ab} = |(r_{ia} - r_{ib})\, \delta_{ij}\, (r_{ja} - r_{jb})|^{\frac{1}{2}}, \tag{8.89}$$

and which is manifestly invariant under the familiar rotational symmetry O(3), the Euclidean symmetry E(3) and the Galilei symmetry G(3.1).

Consider now the complete extension of the above system into the assigned form (1.1). The identification of the isosymmetries requires the isorepresentation of the complete system via Hamilton-Santilli equations (7.32) according to the techniques of Sect. 7. This identifies the isounit $\hat{1}_2$ (or isotopic element $\hat{T}_2$). In turn, the isounit $\hat{1}_2$ permits the construction of the isotopes $\hat{O}(3)$, $\hat{E}(3)$ and $\hat{G}(3.1)$ via the use of the original generators and parameters of O(3), E(3) and G(3.1), respectively, according to the techniques of Sect. 6.

. The system then results to be invariant under the isotopic symmetries $\hat{O}(3)$, $\hat{E}(3)$ and $\hat{G}(3.1)$ when the Hamiltonian is an isoscalar in $T^*\hat{E}_2(r,\hat{\delta},\hat{\Re})$, i.e., of the form

$$H = T(p) + V(r) = p^{\hat{2}} / 2m + V(r), \tag{8.88a}$$

$$p^{\hat{2}} = p_i\, T_{2\,ij}\, p_j, \qquad r = |r_i\, T_{2\,ij}\, r_j|^{\frac{1}{2}}, \tag{8.88b}$$

Similar results hold under isorelativistic extension (Santilli (1988c), (1991d)), yielding the isosymmetries under the isotopes $\hat{O}(3.1)$ and $\hat{P}(3.1)$ of the Lorentz group O(3.1) and of the Poincare' group P(3.1), respectively. See the appendices for more details.

It is easy to see that Santilli's isosymmetries outlined above are not only nonlinear, but also nonlocal, owing to the appearance of the isounit $\hat{1}_2$ directly in their infinitesimal structure. As such, they are indeed the symmetries of the most general known integro-differential systems of ordinary differential equations.

To summarize, in the preceding review, we have outlined the following methodological foundations of Santilli's isotopic liftings of Galilei's, Einstein's special and Einstein's general relativities for the interior dynamical problem:

1) Santilli's basic isofield of the reals $\hat{\Re} = \Re\,\hat{1}$;

2) Santilli's fundamental carrier spaces, the isoeuclidean spaces $\hat{E}(r,\hat{\delta},\hat{\Re})$, the isominkowski spaces $\hat{M}(x,\hat{\eta},\hat{\Re})$, and the isoriemannian spaces $\hat{R}(x,\hat{g},\hat{\Re})$, which provide a geometrization of the interior physical media;

3) Santilli's isotransformation theory, which results to be isolinear and isolocal in the isospaces, but nonlinear and nonlocal when projected in the original spaces;

4) Santilli's isotopic generalization of the conventional Lie's theory (universal enveloping isoassociative algebras, Lie-isotopic algebras and Lie-isotopic groups) characterizing the generalized algebraic structure of the isotopic relativities;

5) The Birkhoff-Santilli mechanics, and its Hamilton-Santilli particularization, characterizing the generalized analytic structure of the isotopic relativities;

6) Santilli's isoanalytic representation of systems (1.1) via the isotopic equations (8.83);

7) Santilli's isosymmetries and methods for their construction from the given equations of motion (1.1).

These advances, however, even though rather considerable, were still considered insufficient by Santilli for the construction of the new generation of covering relativities in the needed depth. In fact, all relativities are an ultimate, symbiotic manifestation of algebraic, analytic and geometric formulations. We have outlined until now the isotopic generalizations of conventional algebraic and analytic structures, but the isotopies of conventional symmetries have not been considered so far.

In his remarkable series of mathematical and physical discoveries Santilli therefore passed to the study of what appear to be his most significant achievements: the isotopic generalizations of the conventional symplectic geometry, affine geometry and Riemannian geometry, which are reviewed in the remaining sections of this volume.

I.9: ISOSYMPLECTIC GEOMETRY

In this section we shall review a nonlocal-integral generalization of the symplectic geometry introduced, apparently for the first time, by Santilli (1988a, b), (1991b, d) under the name of *symplectic-isotopic geometry* , and which is today called *Santilli's isosymplectic geometry* . We shall then show that such a generalized geometry is indeed the geometric counterpart of the Lie-Santilli's theory and of the Birkhoff-Santilli mechanics. We shall finally show that the generalized geometry is indeed applicable for the "direct representation" of nonlinear, nonlocal and nonhamiltonian systems (1.1), that is, their representation directly in the local coordinates of the experimenter.

To avoid a prohibitive length, in this section we shall merely review the main lines of the new geometry, and refer the reader to the quoted literature for

applications.

The literature in the symplectic geometry is rather vast indeed. A list of references can be found in Santilli (1982a), p. 77. In the following, we shall review only the most essential aspects of the conventional symplectic geometry needed for our analysis by following Abraham and Marsden (1967). The literature in the calculus of exterior forms is also so vast to discourage an outline. In this section we shall follow the monograph in the field by Lovelock and Rund (1975).

All quantities considered are assumed to verify the needed continuity conditions, e.g., of being of Class $\mathcal{C}^{\infty}$, which shall be hereon omitted for brevity. Similarly, all neighborhoods of given points are assumed to be star-shaped, or have a similar topology also ignored hereon for brevity.

Let $M(\Re)$ be an n-dimensional (abstract) manifold over the reals $\Re$ and let $T^*M(\Re)$ be its cotangent bundle. We shall denote with $T^*M_1(\Re)$ the manifold $T^*M(\Re)$ equipped with the *canonical one-form* θ defined by (see, e.g., Abraham and Marsden (1967))

$$\theta : T^*M_1(\Re) \Rightarrow T^*(T^*M_1(\Re)), \quad \theta \in \Lambda_1(T^*M_1(\Re)). \tag{9.1}$$

The *fundamental (canonical) symplectic form* is then given by the two-form

$$\omega = d\theta, \tag{9.2}$$

which is nowhere degenerated, exact and therefore closed, i.e., such that $d\omega \equiv 0$. The manifold $T^*M(\Re)$, when equipped with the symplectic two-form ω becomes an (exact) *symplectic manifold* $T^*M_2(\Re)$ in canonical realization. The *symplectic geometry* is the geometry of symplectic manifolds as characterized by exterior forms, Lie's derivative, etc.

Let H be a function on $T^*M_2(\Re)$ called the *Hamiltonian*. A vector-field X on $T^*M_2(\Re)$ is called a *Hamiltonian vector-field* when it verifies the condition

$$X \,^\circ\, \omega = - dH. \tag{9.3}$$

The above equation provides a global, coordinate-free characterization of the conventional Hamilton's equations (those without external terms) for the case of *autonomous systems* , i.e., systems without an explicit dependent in the independent variable (time t).

Finally, we recall that the *Lie derivative* of a vector-field Y with respect to

the vector field X on $T^*M_2(\Re)$ can be defined by

$$L_X Y = [X,Y], \tag{9.4}$$

where [X,Y] is the canonical commutator.

The case of *nonautonomous systems* (those with an explicit dependence on time) requires the further extension to the *contact geometry* (see, e.g., Abraham and Marsden (*loc. cit.*)), and it will not be considered here for brevity because it does not affect the Lie content of the geometry of primary interest for this study.

The Birkhoffian generalization of the above canonical geometry is straightforward, and was worked out in Santilli (1978a) and (1982a).

Introduce in the same cotangent bundle $T^*M_1(\Re)$ the most general possible one-form Θ, called by Santilli the *Birkhoffian* or *Pfaffian one-form*,

$$\Theta:\ T^*M_1(\Re) \Rightarrow T^*(T^*M_1(\Re)), \qquad \Theta \in \Lambda_1(T^*M_1(\Re)). \tag{9.5}$$

The *Birkhoffian two-form* is then given by

$$\Omega = d\Theta, \tag{9.6}$$

under the condition that it is nowhere degenerate. Ω is exact by construction and therefore closed, that is, symplectic. The manifold $T^*M(\Re)$, when equipped with the two-form Ω, becomes an *exact, Birkhoffian, symplectic manifold* $T^*M_2(\Re)$.

Let B be another function on $T^*M_2(\Re)$ called, also by Santilli, the *Birkhoffian*. Then, a non-Hamiltonian vector-field $\hat{X}$ on $T^*M_2(\Re)$ is called a *Birkhoffian vector-field* when it verifies the property

$$\hat{X} \rfloor \Omega = - dB. \tag{9.7}$$

which provides a global, coordinate-free characterization of Birkhoff's equations for autonomous systems.

Similarly, we recall that the *Lie-Santilli derivative* of a vector-field $\hat{Y}$ with respect to a *nonhamiltonian* vector field $\hat{X}$ (Santilli (1982a), p.88) can be written

$$\hat{L}_{\hat{X}}\hat{Y} = [\hat{X} \hat{,} \hat{Y}], \tag{9.8}$$

where the brackets are now Birkhoffian (see below for the explicit form).

The realization of the above global structures in local coordinates is

straightforward. Interpret the space $M(\Re)$ as an Euclidean space $E(r,\Re)$ with local coordinates $r = (r_i)$, $i= 1, 2, ..., n$. Then, the cotangent bundle T^*M becomes $T^*E(r,\Re)$ with local coordinates $(r,p) = (r_i, p_i)$, where $p = dr/dt$ represents the tangent vectors, and we ignore for simplicity of notation the distinction between contravariant and covariant indices in Euclidean spaces (but not in other spaces). The canonical one-form (9.1) then admits the local realization

$$\theta = p_i \, dr_i. \tag{9.9}$$

The Hamiltonian two-form (9.2) admits the realization

$$\omega = d\theta = dp_i \wedge dr_i, \tag{9.10}$$

from which one can easily verify that $d\omega \equiv 0$. A vector-field can then be written

$$X = A_i(r,p)\,\partial / \partial r_i + B_i(r,p)\,\partial / \partial p_i, \tag{9.11a}$$

$$A_i \, dr_i + B_i \, dp_i = -dH, \tag{9.11b}$$

which can hold iff Hamilton's equations are verified, i.e.,

$$\frac{dr_i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = - \frac{\partial H}{dr_i}, \tag{9.12}$$

Finally, Lie's derivative (9.4) admits the simple realization

$$L_X Y = [X,Y] = \frac{\partial X}{\partial r_i}\frac{\partial Y}{\partial p_i} - \frac{\partial Y}{\partial r_i}\frac{\partial X}{\partial p_i}, \tag{9.13}$$

where one recognizes in the commutator the familiar Poisson brackets (Sect. 7).

The realization of the Birkhoffian generalization of the above structures requires the introduction of the unified notation introduced in Eq.s (7.13), i.e.,

$$a = (a^\mu) = (r, p) = (r_i, p_i), \quad \mu = 1, 2, ..., 2n, \quad i = 1, 2, ..., n, \tag{9.14}$$

where we preserve the distinction between contravariant and covariant indices of

the a-coordinates of the cotangent bundle. The canonical one-form can then be written

$$\theta = R^\circ_\mu \, da^\mu \equiv p_i \, dr_i, \quad R^\circ = (p, 0), \tag{9.15}$$

and Hamiltonian two-form (9.10) becomes

$$\omega = d\theta = \tfrac{1}{2} \omega_{\mu\nu} \, da^\mu \wedge da^\nu \equiv dp_i \wedge dr_i, \tag{9.16}$$

where $\omega_{\mu\nu}$ is the *covariant, canonical, symplectic tensor* (7.15), i.e.,

$$(\omega_{\mu\nu}) = \left(\frac{\partial R^\circ_\nu}{\partial a^\mu} - \frac{\partial R^\circ_\mu}{\partial a^\nu} \right) = \begin{pmatrix} 0_{n\times n} & -I_{n\times n} \\ I_{n\times n} & 0_{n\times n} \end{pmatrix} \tag{9.17}$$

A vector-field can then be written

$$X = X_\mu(a) \, \partial / \partial a^\mu. \tag{9.18}$$

The conditions for a Hamiltonian vector-field become

$$\omega_{\mu\nu} \, X^\mu \, da^\mu = -dH, \tag{9.19}$$

and can hold iff

$$X = X_\mu \frac{\partial}{\partial a^\mu} = \omega^{\mu\nu} \frac{\partial H}{\partial a^\nu} \frac{\partial}{\partial a^\mu}, \tag{9.20}$$

where

$$\omega^{\mu\nu} = (|\omega_{\alpha\beta}|^{-1})^{\mu\nu}, \tag{9.21}$$

namely, iff Hamilton's equations (9.12) hold, which in the unified notation can be written as in Eq.s (7.18), i.e.,

$$\dot{a}^\mu = \omega^{\mu\nu} \frac{\partial H}{\partial a^\nu}. \tag{9.22}$$

∂a^{ν}

Finally, Lie's derivative becomes, in unified notation,

$$L_X Y = [X,Y] = \frac{\partial X}{\partial a^{\mu}} \omega^{\mu\nu} \frac{\partial Y}{\partial a^{\nu}}, \tag{9.23}$$

The transition to the Birkhoffian realization is now straight-forward (Santilli (*loc. cit.*)). In fact, it merely requires the transition from the canonical quantities $R^{\circ}(a) = (p, 0)$ to arbitrary quantities $R(a)$ on $T^*E_I(r,\Re)$ under which the Birkhoffian one-form (9.5) assumes the realization

$$\Theta = R_{\mu}(a)\, da^{\mu}, \tag{9.24}$$

while the Birkhoffian two-form (9.6) becomes

$$\Omega = d\Theta = \tfrac{1}{2}\, \Omega_{\mu\nu}(a)\, da^{\mu} \wedge da^{\nu}. \tag{9.25}$$

where $\Omega_{\mu\nu}$ is the *(covariant) symplectic Birkhoff's tensor* (7.3), i.e.

$$\Omega_{\mu\nu}(a) = \frac{\partial R_{\nu}}{\partial a^{\mu}} - \frac{\partial R_{\mu}}{\partial a^{\nu}} \tag{9.26}$$

A Birkhoffian vector-field $\hat{X}$ can no longer be decomposed in the simple form (9.11), but can be written

$$\hat{X} = X^{\mu}\, \partial / \partial a_{\mu}. \tag{9.27}$$

The conditions for a vector-field $\hat{X}$ to be Birkhoffian, Eq.s (9.7), then become

$$\hat{X} \cdot \Omega = \Omega_{\mu\nu}\, \hat{X}^{\nu} da^{\mu} = -dB, \tag{9.28}$$

and they hold iff

$$\hat{X} = \hat{X}^{\mu} \frac{\partial}{\partial a^{\mu}} = \Omega^{\mu\nu} \frac{\partial B}{\partial a^{\nu}} \frac{\partial}{\partial a^{\mu}}, \tag{9.29}$$

where

$$\Omega^{\mu\nu} = (\,|\,\Omega_{\alpha\beta}|^{-1})^{\mu\nu}, \tag{9.30}$$

which can hold iff the autonomous Birkhoff's equations (Birkhoff (1927)), Eq.s (7.4), hold, i.e.,

$$\dot{a}^{\mu} = \hat{X}^{\mu} = \Omega^{\mu\nu}(a)\frac{\partial B(a)}{\partial a^{\nu}}. \tag{9.31}$$

Similarly, the Lie-Santilli derivative (9.8) assumes the realization

$$\hat{L}_{\hat{X}}\hat{Y} = [\hat{X}\,\hat{,}\,\hat{Y}] = \frac{\partial \hat{X}}{\partial a^{\mu}}\,\Omega^{\mu\nu}(a)\,\frac{\partial \hat{Y}}{\partial a^{\nu}}, \tag{9.32}$$

For additional aspects, the reader may consult Santilli (1982a), the appendices of Ch. 4.

Note that an arbitrary vector-field $\hat{X}$ is not Hamiltonian in a given local chart. A central result of Santilli (*loc. cit.*) can be reformulated as follows

THEOREM 9.1 (DIRECT UNIVERSALITY OF THE SYMPLECTIC GEOMETRY FOR LOCAL NEWTONIAN SYSTEMS): An arbitrary, local-differential, analytic and regular vector-field $\hat{X}$ on a given chart on $T^*M_2\,(r,\Re)$ always admits a direct representation as a Birkhoffian vector-field, i.e., a representation directly in the chart considered.

The physical implications are the following. When considering conservative-potential systems of the exterior dynamical problem (Sect. 1), the vector-fields are evidently Hamiltonian in the frame of the experimenter. However, when considering the nonconservative systems of the interior dynamical problem, the vector-fields are generally nonhamiltonian in the frame of the experimenter[15].

[15] It is appropriate here to recall that the general lack of Hamiltonian character of vector-fields is precisely the reason which lead Lagrange and Hamilton to formulate their analytic equations *with external terms* (see Sect. 1 for details and historical references). Note that Hamilton's equations with external terms are outside the representational capabilities of the symplectic geometry. Nevertheless, the Birkhoffian realization of the symplectic geometry essentially allows the representation of these historical external terms, when of local-differential character, without abandoning the symplectic character of the geometry, but by

Now, under sufficient topological conditions, the *Lie-Koening theorem* ensures that a nonhamiltonian vector-field can always be transformed into a Hamiltonian form under a suitable change of coordinates (see the analytic and geometric proofs of Santilli (*loc. cit.*)).

However, since the original vector-field is nonhamiltonian by assumption, the transformations must necessarily be *noncanonical* and *nonlinear*, thus creating evident physical problems, e.g., conventional relativities become inapplicable because turned into *noninertial* formulations.

This creates the need of Santilli's "direct representation" of the physical systems considered, that is, their representation, first, in the frame of the experimenter, as per Theorem 9.1. Once this basic task is achieved, then the judicious use of the transformation theory may have some physical value.

Intriguingly, the identification of the Lie-Koening transformation $a \Rightarrow a'$ turning nonhamiltonian systems $\hat{X}(a)$ into Hamiltonian forms $\hat{X}(a(a')) = X(a')$, directly implies the Birkhoffian representation of Theorem 9.1 in the a-frame of the observer. In fact, Birkhoff's equations (9.31) in the a-frame can be characterized precisely via a *noncanonical* transformation $a' \Rightarrow a$ of Hamilton's equations (9.22) in the a'-frame, i.e.,

$$\omega_{\mu\nu}\dot{a}'^{\nu} - \frac{\partial H(a')}{\partial a'^{\mu}} = \frac{\partial a^{\rho}}{\partial a'^{\mu}} [\Omega_{\rho\sigma}(a) - \frac{\partial B(a)}{\partial a^{\rho}}] = 0, \tag{9.33a}$$

$$H(a'(a)) = B(a), \tag{9.33b}$$

(see Santilli (*loc. cit.*), p. 130 for details).

We are now sufficiently equipped to review Santilli's isosymplectic geometry. To begin, let us recall that the geometry outlined above is strictly local-differential. In particular, the vector-fields cannot incorporate nonlocal-integral terms without the construction of a suitable, rather complex revision of the geometry via an appropriate nonlocal-integral topology.

Santilli's new geometry is essentially the generalization of the symplectic geometry into a nonlocal-integral form which is mathematically simple and physically effective, as well as permitting the direct representation of vector-fields with nonlocal-integral components.

For this purpose, Santilli (1988b), (1991b) first rewrites the canonical realization of the symplectic geometry in the following way. Consider again the original,

assuming instead the most general possible exact symplectic two-forms (Santilli (1982a)). The isosymplectic geometry is the final expression allowing the representation of the external terms also when of nonlocal-integral type (Santilli (1988b), (1991a)).

abstract cotangent bundle $T^*M(\mathfrak{R})$, and let

$$I^\circ = (I_{n\times n}) = \mathrm{diag.}(1, 1, \ldots, 1) \equiv T^{\circ -1} \tag{9.34}$$

be its unit. Then, the canonical one form (9.1) can be identically written in terms of the factorization

$$\theta \equiv \hat{\theta}^\circ = \theta \times T^\circ : T^*\hat{M}_I{}^\circ \Rightarrow T^*(T^*\hat{M}_I{}^\circ), \tag{9.35}$$

while the canonical two-form (9.2) becomes

$$\omega \equiv \hat{\omega}^\circ = d\hat{\theta}^\circ = (d\theta)\times T + \theta\, dT \equiv \omega \times T^\circ \tag{9.36}$$

This implies that, in the realization $T^*E(r,\mathfrak{R})$ of $T^*M(\mathfrak{R})$ with local chart $a = (r, p)$, we can exhibit the isotopic element, this, time given by the trivial identity T°, directly in the canonical-symplectic tensor

$$\hat{\omega}^\circ{}_{\mu\nu} = T^\circ{}_\mu{}^\alpha\, \omega_{\alpha\nu}, \tag{9.37}$$

Then, its contravariant version, the unit I°, is exhibited in the Lie-tensor of the theory,

$$\hat{\omega}^{\circ\mu\nu} = \omega^{\mu\alpha}\, I^\circ{}_\alpha{}^\nu. \tag{9.38}$$

The main idea of Santilli's isosymplectic geometry is that of reaching a generalization of two-form (9.38) in which the trivial isotopy is replaced by the most general possible isotopy, i.e.,

$$\hat{\omega}_{\mu\nu} = T_\mu{}^\alpha(a)\, \omega_{\alpha\nu}, \tag{9.39}$$

under the conditions of characterizing an exact and therefore closed two-form.

In this way, the conventional, local-differential, topological structure of the symplectic geometry is preserved in its entirety in the canonical two-form ω, while all nonlocal-integral terms can be incorporated in the isotopic element T.

The corresponding algebraic tensor is then of the type

$$\hat{\omega}^{\mu\nu} = \omega^{\mu\alpha}\, 1_\alpha{}^\nu(a), \tag{9.40}$$

namely, it is precisely of the Lie–isotopic type with the explicit identification of the isounit directly in the structure of the Lie product, as desired for this study.

The topological consistency of the geometry then follows from that of the underlying Lie–Santilli algebra discussed earlier.

For clarity as well as for ready comparison of the results, Santilli followed the presentation of the conventional exterior calculus by Lovelock and Rund (1975), by preserving their notation, and the same will be done in this volume. A generic 2n-dimensional bundle will therefore be denoted $T^*M(\mathfrak{R})$ with generic local chart x = (x^i), i = 1, 2, ..., 2n. We shall return to our a–coordinates later on for specific physical interpretations.

To begin, let us submit the manifold $M(\mathfrak{R})$ to one of the infinitely possible isotopic liftings into n–dimensional isospaces $\hat{M}(\hat{\mathfrak{R}})$ over the isofields $\hat{\mathfrak{R}}$, and let $T^*\hat{M}(\hat{\mathfrak{R}})$ be its *"isocotangent bundle"*, that is, the conventional bundle only referred to isospace $\hat{M}$. Introduce one of the infinitely possible, symmetric, nonsingular and real–valued isounits of $\hat{\mathfrak{R}}$ in the original charts x

$$\hat{1} = \hat{1}(x) = (\hat{1}^i_{\ j}) = (\hat{1}_j^{\ i}) = (\hat{1}^j_{\ i}) = (\hat{1}_i^{\ j}) = T^{-1} \tag{9.41a}$$

$$T = T(x) = (T^i_{\ j}) = (T_j^{\ i}) = (T^j_{\ i}) = (T_i^{\ j}). \tag{9.41b}$$

For mathematical consistency (e.g., to preserve isolinearity, see Sect. 4), conventional linear transformations on $T^*M(\mathfrak{R})$, e.g.,

$$x' = Ax, \quad \text{or} \quad x'^i = A^i_{\ j}\, x^j, \tag{9.42}$$

must be necessarily generalized on $T^*\hat{M}(\hat{\mathfrak{R}})$ into Santilli's isotransformations

$$x' = A{*}x, \quad \text{or} \quad x' = A^i_{\ r}\, T^r_{\ s}\, x^s. \tag{9.43}$$

In the conventional case, the differentials dx and dx′ of the two coordinate systems are related by the familiar expressions

$$dx' = A\, dx, \quad \text{or} \quad dx'^i = A^i_{\ j}\, dx^j, \tag{9.44}$$

with the realization, say, for the coordinate transformations $x \Rightarrow x' = x'(x)$
$\partial x'$ ∂x^i

$$dx' = \frac{}{\partial x} dx, \quad \text{or} \quad d\bar{x}^i = \frac{}{\partial x^j} dx^j. \tag{9.45}$$

However, the same notion of differentials dx and dx' becomes inconsistent in the isocontangent bundle $T^*\hat{M}(\hat{\Re})$. Santilli (*loc. cit.*) therefore introduced the generalized notion of *isodifferentials* $\hat{d}x$ and $\hat{d}\bar{x}$ which hold when interconnected by the isotopic laws

$$\hat{d}\bar{x} = A * \hat{d}x, \quad \text{or} \quad \hat{d}\bar{x}^i = A^i{}_r T^r{}_s \hat{d}x^s, \tag{9.46}$$

with the particular realization, say, for the case of the isotransformations $x \Rightarrow \bar{x}(x)$

$$\hat{d}\bar{x} = \frac{\partial \bar{x}}{\partial x} * \hat{d}x, \quad \text{or} \quad \hat{d}\bar{x}^i = \frac{\partial \bar{x}^i}{\partial x^r} T^r{}_s \hat{d}x^s. \tag{9.47}$$

The full geometrical meaning of the above isotransformations and of the isodifferential $\hat{d}x$, will be evident later on when studying notions such as Santilli's isoparallel transport and isogeodesics. At this moment we shall simply assume the notions and derive their consequences.

Let $\phi(x)$ be an *isoscalar function* on $T^*\hat{M}(\hat{\Re})$. Then its isodifferential is given by

$$\hat{d}\phi = \frac{\partial \phi}{\partial x} * \hat{d}x, \quad \text{or} \quad \hat{d}\phi(x) = \frac{\partial \phi}{\partial x^r} T^r{}_s \hat{d}x^s. \tag{9.48}$$

where the partial derivative is the conventional one.

Similarly, Santilli defines $X = (X^i)$ as a contravariant *isovector-field* on $T^*\hat{M}(\hat{\Re})$, that is, an ordinary vector-field although defined on an isospace. Then its isodifferential is given by

$$\hat{d}X = \frac{\partial X}{\partial x} * \hat{d}x, \quad \text{or} \quad \hat{d}X^i = \frac{\partial X^i}{\partial x^r} T^r{}_s \hat{d}x^s, \tag{9.49}$$

Thus, an isovector-field on $T^*\hat{M}(\hat{\Re})$ transforms according to the isotopic laws

$$\bar{X}(\bar{x}) = \frac{\partial \bar{X}}{\partial x} * X(x), \quad \text{or} \quad \bar{X}^i(\bar{x}) = \frac{\partial \bar{X}^i}{\partial x^r} T^r{}_s(x)\, X^s(x). \tag{9.50}$$

Note that, while for conventional transformations (9.42) on $T^*M(x,\Re)$ we have $\partial x'/\partial x = A$, we now have for isotransformations (9.43)

$$\frac{\partial \bar{x}^i}{\partial x^j} = A^i_r T^r_j + A^i_r \frac{\partial T^r_s}{\partial x^j} x^s. \tag{9.51}$$

By using the above results and the usual chain rule for partial differentiation, one easily gets from law (9.51)

$$\frac{\partial \bar{X}^j}{\partial \bar{x}^k} = \frac{\partial^2 \bar{x}^j}{\partial x^s \partial x^i} \frac{\partial x^s}{\partial \bar{x}^k} T^i_r X^r + \frac{\partial \bar{x}^j}{\partial x^i} \frac{\partial x^s}{\partial \bar{x}^k} Tir \frac{\partial X^r}{\partial x^s} =$$

$$+ \frac{\partial \bar{x}^j}{\partial x^i} \frac{\partial x^s}{\partial \bar{x}^k} \frac{\partial T^i_r}{\partial x^s} X^r. \tag{9.52}$$

One can see in this way that, in addition to the isotopy of the conventional two terms of this expression (see Eq.s (3.5), p. 67, Lovelock and Rund, (*loc., cit.*)), Santilli obtains an additional third term. Note that the quantity $\partial \bar{X}_j / \partial \bar{x}^k$ is not a mixed tensor of rank (1.1), exactly as it happens in the conventional case.

From the preceding results one can then compute the isodifferential of a contravariant isovector-field

$$\hat{d} \bar{X}^j = \frac{\partial \bar{X}_j}{\partial \bar{x}^k} T^k_r \hat{d}\bar{x}^r =$$

$$= \frac{\partial^2 \bar{x}^j}{\partial x^s \partial x^i} T^i_r X^r \hat{d}x^s + \frac{\partial \bar{x}^j}{\partial x^i} T^i_r \frac{\partial X^r}{\partial x^s} \hat{d}x^s + \frac{\partial \bar{x}^j}{\partial x^i} \frac{\partial T^i_r}{\partial x^s} X^r \hat{d}x^s \tag{9.53}$$

A *contravariant isotensor* X^{ij} of rank two on $\hat{M}(\hat{\Re})$ is evidently characterized by the transformation laws

$$\bar{X}^{(2)}(\bar{x}) = \frac{\partial \bar{x}}{\partial x} * \frac{\partial \bar{x}}{\partial x} * X^{(2)}(x), \quad \bar{X}^{ij}(\bar{x}) = \frac{\partial \bar{x}^i}{\partial x^r} T^r_p \frac{\partial \bar{x}^j}{\partial x^s} T^s_q X^{pq}(x), \tag{9.54}$$

Similar extensions to higher orders, as well as to contravariant isotensors of

rank (0.s) and to generic tensors of rank (r.s) are left as an exercise for the interested reader.

By following Santilli, all preceding expressions (9.42)–(9.54) have been written in both, the abstract form and their realization in local coordinates, to illustrate that the notion of isotransformations and isodifferentials do constitute isotopies, in the sense that all distinctions between conventional and isotopic notions cease to exist at the abstract, realization-free level.

Santilli then introduces the notion of *one-isoform* on $T^*\hat{M}_I(\hat{\Re})$ as the quantity

$$\Phi_I = A \hat{*} \hat{\partial} x = A_i \, T^i{}_j \, \hat{\partial} x^j. \tag{9.55}$$

and studies the algebraic operations of isodifferentials and one-isoforms. The sum of two one-isoforms is the conventional sum. In fact, given two one-isoforms $\Phi_I{}^1 = A \hat{*} \hat{\partial} x$ and $\Phi_I{}^2 = B \hat{*} \hat{\partial} x$, their sum is given by

$$\Phi_I{}^1 + \Phi_I{}^2 = (A + B) \hat{*} \hat{\partial} x. \tag{9.56}$$

The isoproduct of one-isoform $\Phi_I = A \hat{*} \hat{\partial} x$ with an isonumber $\hat{n} \in \hat{\Re}$ is the conventional product,

$$\hat{n} \hat{*} \Phi_I \equiv n \, \Phi_I. \tag{9.57}$$

For the product of two or more one-isoforms $\Phi_I{}^k = A^k \hat{*} \hat{\partial} x$, $k = 1, 2, 3, \ldots$ we introduce the *isoexterior*, or *isowedge product* denoted with the symbol $\hat{\wedge}$, which verifies the same axioms of the conventional exterior product, that is, distributive laws and anticommutativity, i.e.

$$(\Phi_I{}^1 + \Phi_I{}^2) \hat{\wedge} \Phi_I{}^3 = \Phi_I{}^1 \hat{\wedge} \Phi_I{}^3 + \Phi_I{}^2 \hat{\wedge} \Phi_I{}^3, \tag{9.58a}$$

$$\Phi_I{}^1 \hat{\wedge} (\Phi_I{}^2 + \Phi_I{}^3) = \Phi_I{}^1 \hat{\wedge} \Phi_I{}^2 + \Phi_I{}^1 \hat{\wedge} \Phi_I{}^3, \tag{9.58b}$$

$$\Phi_I{}^1 \hat{\wedge} \Phi_I{}^2 = - \Phi_I{}^2 \hat{\wedge} \Phi_I{}^1. \tag{9.58c}$$

The product of two one-isoforms $\Phi_I{}^1 = A \hat{*} \hat{\partial} x$ and $\Phi_I{}^2 = B \hat{*} \hat{\partial} x$ shall be called a *two-isoform* on $T^*\hat{M}(x, \hat{\Re})$, and can be written

$$\Phi_{\hat{2}} = \Phi_1{}^1 \hat{\wedge} \Phi_1{}^2 = A_i T^i{}_r B_j T^j{}_s \, \hat{d}x^r \hat{\wedge} \hat{d}x^s =$$

$$= \tfrac{1}{2} (A_i T^i{}_r B_j T^j{}_s - A_i T^i{}_s B_j T^j{}_r) \, \hat{d}x^r \hat{\wedge} \hat{d}x^s =$$

$$= \tfrac{1}{2} A_i B_j (T^i{}_r T^j{}_s - T^i{}_s T^j{}_r) \, \hat{d}x^r \hat{\wedge} \hat{d}x^s, \tag{9.59}$$

thus showing a clear deviations from the conventional exterior calculus (compare with Lovelock and Rund (*loc. cit.*), p. 132).

For the case of the isoexterior product of the one-isoforms Santilli obtains the three-isoform

$$\Phi_3 = \Phi_1{}^1 \wedge \Phi_1{}^2 \hat{\wedge} \Phi_1{}^3, \tag{9.60}$$

$$= A^1{}_{i_1} A^2{}_{i_2} A^3{}_{i_3} \, \delta^{j_1 j_2 j_3}{}_{k_1 k_2 k_3} \, T^{i_1}{}_{j_1} T^{i_2}{}_{j_2} T^{i_3}{}_{j_3} \, \hat{d}x^{k_1} \wedge \hat{d}x^{k_2} \hat{\wedge} \hat{d}x^{k_3},$$

where (see Lovelock and Rund (*loc. cit.*)

$$\delta^{i_1 i_2}{}_{j_1 j_2} = \det. \begin{pmatrix} \delta^{i_1}{}_{j_1} & \delta^{i_1}{}_{j_2} \\ \delta^{i_2}{}_{j_1} & \delta^{i_2}{}_{j_2} \end{pmatrix} \tag{9.61a}$$

$$\delta^{i_1 i_2 i_3}{}_{j_1 j_2 j_3} = \det. \begin{pmatrix} \delta^{i_1}{}_{j_1} & \delta^{i_1}{}_{j_2} & \delta^{i_1}{}_{j_3} \\ \delta^{i_2}{}_{j_1} & \delta^{i_2}{}_{j_2} & \delta^{i_2}{}_{j_3} \\ \delta^{i_3}{}_{j_1} & \delta^{i_3}{}_{j_2} & \delta^{i_3}{}_{j_3} \end{pmatrix}, \tag{9.61b}$$

etc. The extension to n-isoforms on $T^*\hat{M}(\hat{\Re})$ is left to the interested reader.

Given n one-isoforms $\Phi_1{}^k = A^k * \hat{d}x$, $k = 1, 2, \ldots n$, they are said to be *isolinearly dependent* when

$$\Phi_1{}^1 \hat{\wedge} \ldots\ldots \hat{\wedge} \Phi_1{}^n \equiv 0. \tag{9.62}$$

Note that given n one-isoforms linearly dependent on $M(x,\Re)$, their isotopic images are not necessarily dependent.

Evidently, in an n-dimensional isomanifold $\hat{M}(\hat{\Re})$ there exist a maximum of n linearly independent one-isoforms as in the conventional case, with basis $\hat{d}x^1$, ..., $\hat{d}x^n$. The space $\hat{M}(\hat{\Re})$ equipped with iso-oneforms is the cotangent space $T^*\hat{M}_1(\hat{\Re})$ at a given point.

Similarly, two-isoforms are elements of an isomanifold here denoted $T^*\hat{M}_2(\hat{\Re})$ of $\frac{1}{2}n(n-1)$-dimension with basis $\hat{d}x^i \hat{\wedge} \hat{d}x^j$, $i < j$, as in the conventional case. A similar situation occurs for p-isoforms

$$\Phi_p = A_{i_1 i_2 \ldots i_p} T^{i_1}{}_{j_1} T^{i_2}{}_{j_2} \times\times\times T^{i_p}{}_{j_p} \hat{d}x^{j_1} \hat{\wedge} \hat{d}x^{j_2} \hat{\wedge} \ldots \hat{\wedge} \hat{d}x^{j_p}, \tag{9.63}$$

and related isomanifolds $T^*\hat{M}_p(\hat{\Re})$.

As an incidental note we point out without treatment the *Grassmann-isotopic algebra* $\hat{G}$ or *isograssmann algebra*, which is given by the direct sum

$$\hat{G} = \sum_{k=0,1,2,\ldots,n} T^*\hat{M}_k(\hat{\Re}). \tag{9.64}$$

The necessary and sufficient conditions for a two-isoform (9.59) to be identically null are that

$$\delta^{i_1 i_2}{}_{j_1 j_2} A^1{}_{k_1} A^2{}_{k_2} T^{k_1}{}_{i_1} T^{k_2}{}_{i_2}$$

$$= A^1{}_{k_1} A^2{}_{k_2} (T^{k_1}{}_{i_1} T^{k_2}{}_{i_2} - T^{k_1}{}_{i_2} T^{k_2}{}_{i_1}) \equiv 0. \tag{9.65}$$

A similar situation occurs for p-isoforms.

We now study the differential calculus that is applicable to p-isoforms. Let $\Phi_1 = A{*}\hat{d}x$ be a one-isoform. Santilli introduces the *isoexterior derivative* of Φ_1 (also called *isoexterior differential*) and denoted with $\hat{d}\Phi_1$, as the two-isoform

$$\Phi_{\hat{2}} = \hat{d}\Phi_1 = \frac{\partial (A_{i_1} T^{i_1}{}_{j_1})}{\partial x^{i_2}} T^{i_2}{}_{j_2} \hat{d}x^{j_1} \hat{\wedge} \hat{d}x^{j_2} = \tag{9.66}$$

$$= \Big(\frac{\partial A_{i_1}}{\partial x^{i_2}} T^{i_1}{}_{j_1} T^{i_2}{}_{j_2} + A_{i_1} \frac{\partial T^{i_1}{}_{j_1}}{\partial x^{i_2}} T^{i_2}{}_{j_2} \Big) \hat{d}x^{j_1} \hat{\wedge} \hat{d}x^{j_2} =$$

$$= \tfrac{1}{2}\delta^{j_1 j_2}_{k_1 k_2} \left(\frac{\partial A_{i_1}}{\partial x^{i_2}} T^{i_1}{}_{j_1} T^{i_2}{}_{j_2} + A_{i_1} \frac{\partial T^{i_1}{}_{j}}{\partial x^{i_2}} T^{i_2}{}_{j_2} \right) \hat{d}x^{k_1} \hat{\wedge} \hat{d}x^{k_2}$$

from which one can see that $\hat{d}\Phi_1$ is no longer the curl of the vector field A_{i_1}, but something more general, although admitting the conventional formulation as a particular case for $\hat{1} = 1$.

The isoexterior derivative of a two-isoform

$$\Phi_2 = A_{i_1 i_2} T^{i_1}{}_{j_1} T^{i_2}{}_{j_2} \hat{d}x^{j_1} \hat{\wedge} \hat{d}x^{j_2}, \tag{9.67}$$

is given by the three-isoform

$$\Phi_3 = \hat{d}\Phi_2 = \left(\frac{\partial A_{i_1 i_2}}{\partial x^{i_3}} T^{i_1}{}_{j_1} T^{i_2}{}_{j_2} T^{i_3}{}_{j_3} + A_{i_1 i_2} \frac{\partial T^{i_1}{}_{j_1}}{\partial x^{i_3}} T^{i_2}{}_{j_2} T^{i_3}{}_{j_3} + \right.$$

$$\left. + A_{i_1 i_2} T^{i_1}{}_{j_1} \frac{\partial T^{i_2}{}_{j_2}}{\partial x^{i_3}} T^{i_3}{}_{j_3} \right) \hat{d}x^{j_1} \hat{\wedge} \hat{d}x^{j_2} \hat{\wedge} \hat{d}x^{j_3}. \tag{9.68}$$

It is easy to see that the isoexterior derivative of the isoexterior product of a p-isoform Φ_p and a q-isoform Φ_q is given by

$$\hat{d}(\Phi_p \hat{\wedge} \Phi_q) = (\hat{d}\Phi_p) \hat{\wedge} \Phi_q + (-1)^p \Phi_p \hat{\wedge} (\hat{d}\Phi_q). \tag{9.69}$$

Santilli then an *isoexact p-isoform* $\hat{\Phi}_p$ when there exists a (p-1) form $\hat{\Phi}_{p-1}$ such that

$$\hat{\Phi}_p = \hat{d}\, \hat{\Phi}_{p-1}. \tag{9.70}$$

and *isoclosed p-isoform* $\hat{\Phi}_p$ when

$$\hat{d}\, \Phi_p = 0. \tag{9.71}$$

A most significant property of the isosymplectic geometry can be expressed as follows.

LEMMA 9.1 (Poincaré–Santilli Lemma; Santilli (1988b), 1991b)): Under sufficient regularity and continuity conditions, the Poincaré Lemma admits an infinite number of isotopic images, i.e., given an exact p-form $\Phi_p = d\Phi_{p-1}$, there exists an infinite number of isotopies of Φ_{p-1} into isoforms $\hat{\Phi}_{p-1}$

$$\Phi_{p-1} \Rightarrow \hat{\Phi}_{p-1}, \tag{9.72}$$

with consequential isotopies of the p-form

$$\Phi_p = d(\Phi_{p-1}) \Rightarrow \hat{\Phi}_p = \hat{d}(\hat{\Phi}_{p-1}), \tag{9.73}$$

for which the isoexterior derivative of the isoexact p-isoforms are identically null,

$$\hat{d}(\hat{d}\,\hat{\Phi}_{p-1}) \equiv 0. \tag{9.74}$$

PROOF: Consider an isoexact two-isoform

$$\hat{\Phi}_2 = \hat{d}\,\hat{\Phi}_1 = \hat{d}(A_i\, T^i{}_j dx^j). \tag{9.75}$$

Then, under the necessary regularity and continuity conditions, its isoexterior derivative

$$\hat{d}\,\hat{\Phi}_2\,(\hat{d}\hat{\Phi}_1) =$$

$$= \Big(\frac{\partial^2 A_{i_1}}{\partial x^{i_2}\,\partial x^{i_3}} T^{i_1}{}_{j_1} T^{i_2}{}_{j_2} T^{i_3}{}_{j_3} + \frac{\partial A_{i_1}}{\partial x^{i_2}} \frac{\partial T^{i_1}{}_{j_1}}{\partial x^{i_3}} T^{i_2}{}_{j_2} T^{i_3}{}_{j_3} +$$

$$+ \frac{\partial A_{i_1}}{\partial x^{i_2}} T^{i_1}{}_{j_1} \frac{\partial T^{i_2}{}_{j_2}}{\partial x^{i_3}} T^{i_3}{}_{j_3}\Big)\, dx^{j_1} \hat{\wedge}\, dx^{j_2} \wedge dx^{j_3}, \tag{9.76}$$

is identically null for all infinitely possible isotopic elements, as the reader can verify via simple but tedious calculations based on the antisymmetrization of all indices. An iteration of the procedure then proves the lemma at any (finite) order p. QED.

In short, the existence of consistent isotopies of the Poincare' Lemma proves the consistency of Santilli's isosymplectic geometry and underlying isoexterior calculus.

The mathematical relevance of Lemma 9.1 is provided by the fact that the abstract, realization-free axioms

$$\Phi_2 = d\Phi_1, \quad d\Phi_2 \equiv 0, \tag{9.77a}$$

$$\Phi_3 = d\Phi_2, \quad d\Phi_3 \equiv 0, \quad \text{etc.} \tag{9.77b}$$

admit the conventional realization based on an ordinary manifold, as well as an infinite number of additional realizations for each given original, conventional form which can be readily identified via Santilli's isomanifolds. The latter realizations are generally inequivalent owing to the generally different isotopic elements or isounits.

While the conventional Poincaré Lemma characterizes the geometric foundations of the Galilei's relativity, Einstein's special relativity and Einstein's gravitation for the exterior dynamical problem, the Poincaré–Santilli Lemma constitutes the geometric characterization of the covering isotopic relativities for the interior problem.

Note that the infinite possibilities of different isotopies (9.77) are geometrically equivalent, but physically inequivalent, in the sense that they characterize corresponding integro-differential systems (1.1) with inequivalent solutions.

We shall now consider some cases of exact isoclosed isoforms. Consider a one-isoform $\hat{\Phi}_1$ on $T^*\hat{M}_1(\hat{\Re})$. Then, $\hat{d}\hat{\Phi}_1 \equiv 0$, iff

$$\tfrac{1}{2}\delta^{j_1 j_2}{}_{k_1 k_2} \left(\frac{\partial A_{i_1}}{\partial x^{i_2}} T^{i_1}{}_{j_1} T^{i_2}{}_{j_2} + A_{i_1} \frac{\partial T^{i_1}{}_{j}}{\partial x^{i_2}} T^{i_2}{}_{j_2} \right) \equiv 0, \tag{9.78}$$

namely, the isoclosure of a one-isoform does not imply that the conventional curl of the vector A is null.

Similarly, given a exact two-isoform $\hat{\Phi}_2 = \hat{\partial}\hat{\Phi}_1$, the property $\hat{\partial}\hat{\Phi}_2 \equiv 0$ holds

iff

$$\delta^{j_1 j_2 j_3}{}_{k_1 k_2 k_3}\left(\frac{\partial^2 A_{i_1}}{\partial x^{i_2}\,\partial x^{i_3}} T^{i_1}{}_{j_1} T^{i_2}{}_{j_2} T^{i_3}{}_{j_3} + \frac{\partial A_{i_1}}{\partial x^{i_2}} \frac{\partial T^{i_1}{}_{j_1}}{\partial x^{i_3}} T^{i_2}{}_{j_2} T^{i_3}{}_{j_3} + \frac{\partial A_{i_1}}{\partial x^{i_2}} T^{i_1}{}_{j_1} \frac{\partial T^{i_2}{}_{j_2}}{\partial x^{i_3}} T^{i_3}{}_{j_3}\right) \equiv 0. \tag{9.79}$$

We now pass to the identification of the isosymplectic geometry. For this purpose, let us review the interplay between exact symplectic two-forms and Lie-Santilli algebras (see Sect. 6). Recall that a conventional two-form on an even, 2n-dimensional manifold $T^*M_2(\Re)$ with covariant-geometric tensor $\Omega_{i_1 i_2}$

$$\Phi_2 = \tfrac{1}{2}\Omega_{i_1 i_2}\, dx^{i_1} \wedge dx^{i_2}, \tag{9.80}$$

characterizes, in its equivalent contravariant version, the algebra brackets among functions $A(x)$ and $B(x)$ on $T^*M_2(\Re)$

$$[A \,\hat{,}\, B] = \frac{\partial A}{\partial x^{i_1}}\, \Omega^{i_1 i_2}\, \frac{\partial B}{\partial x^{i_2}}, \tag{9.81}$$

where the contravariant-algebraic tensor $\Omega^{i_1 i_2}$ is given by the familiar rule

$$\Omega^{i_1 i_2} = (\|\Omega_{j_1 j_2}\|^{-1})^{i_1 i_2}. \tag{9.82}$$

Now, the integrability conditions for two-form (9.80) to be an exact symplectic two-form are given by

$$\Omega_{i_1 i_2} + \Omega_{i_2 i_1} \equiv 0, \tag{9.83a}$$

$$\frac{\partial \Omega_{i_1 i_2}}{\partial x^{i_3}} + \frac{\partial \Omega_{i_2 i_3}}{\partial x^{i_1}} + \frac{\partial \Omega_{i_3 i_1}}{\partial x^{i_2}} \equiv 0, \tag{9.83b}$$

The above conditions are equivalent to the integrability conditions

$$\Omega^{i_1 i_2} + \Omega^{i_2 i_1} \equiv 0, \tag{9.84a}$$

$$\Omega^{i_1 k}\frac{\partial\Omega^{i_2 i_3}}{\partial x^k} + \Omega^{i_2 k}\frac{\partial\Omega^{i_3 i_1}}{\partial x^k} + \Omega^{i_3 k}\frac{\partial\Omega^{i_1 i_2}}{\partial x^k} \equiv 0, \tag{9.84b}$$

for generalized brackets (9.81)) to be Lie-Santilli, i.e., verify the Lie algebra axioms in their most general possible, classical, regular realization on $T^*M_2(\Re)$

$$[A \hat{,} B] + [B \hat{,} A] \equiv 0, \tag{9.85a}$$

$$[[A \hat{,} B] \hat{,} C] + [[B \hat{,} C] \hat{,} A] + [[C \hat{,} A] \hat{,} B] \equiv 0. \tag{9.85b}$$

Thus, the exact character of the two-form $\Phi_2 = d\Phi_1$ implies its closure $d\Phi_2 \equiv 0$ (Poincaré Lemma), which, in turn, guarantees that the underlying brackets are Lie-Santilli, with the canonical case being a trivial particular case (see the analytic, algebraic, and geometric proofs of Santilli (1982a), Sect. 4.1.5).

Santilli has established via Lemma 9.1 that all the above results on the conventional exterior calculus persist under isotopies. A primary purpose of the isosymplectic geometry is then that of identifying the isounit of the Lie-Santilli algebra directly in the structure of the the two-isoform.

DEFINITION 9.1: Under sufficient continuity and regularity conditions, "Santilli's exact isosymplectic manifolds" are 2n-dimensional isomanifolds $T^*\hat{M}_2(x,\hat{\Re})$ equipped with an exact and nowhere degenerate two-isoforms

$$\hat{\Phi}_2 = \tfrac{1}{2}\,\Omega_{i_1 i_2}(x)\, \hat{d}x^{i_1} \hat{\wedge} \hat{d}x^{i_2} = \tag{9.86}$$

$$= \hat{d}\hat{\Phi}_1 = \frac{\partial (A_{i_1} T^{i_1}{}_{j_1})}{\partial x^{i_2}} T^{i_2}{}_{j_2}\, \hat{d}x^{j_1} \hat{\wedge} \hat{d}x^{j_2} =$$

$$= \Big(\frac{\partial A_{i_1}}{\partial x^{i_2}} T^{i_1}{}_{j_1} T^{i_2}{}_{j_2} + A_{i_1} \frac{\partial T^{i_1}{}_{j_1}}{\partial x^{i_2}} T^{i_2}{}_{j_2} \Big)\, \hat{d}x^{j_1} \hat{\wedge} \hat{d}x^{j_2} =$$

$$= \tfrac{1}{2}\,\delta^{j_1 j_2}{}_{k_1 k_2} \Big(\frac{\partial A_{i_1}}{\partial x^{i_2}} T^{i_1}{}_{j_1} T^{i_2}{}_{j_2} + A_{i_1} \frac{\partial T^{i_1}{}_{j}}{\partial x^{i_2}} T^{i_2}{}_{j_2} \Big)\, \hat{d}x^{k_1} \hat{\wedge} \hat{d}x^{k_2}$$

$$\partial x^{i_2} \qquad\qquad \partial x^{i_2}$$

which is such to admit the factorization

$$\hat{\Omega}_{i_1 i_2} = \Omega_{i_1 k} \times T_2{}^k{}_{i_2}, \quad \hat{T}_2 > 0, \tag{9.87}$$

where T_2 is the nowhere degenerate, symmetric and real-valued isotopic element of $T^*\hat{M}_2(x,\hat{\Re})$, and

$$\Omega_{i_1 i_2} = \frac{\partial A_{i_2}}{\partial x^{i_1}} - \frac{\partial A_{i_1}}{\partial x^{i_2}}, \tag{9.88}$$

is Birkhoff's tensor (7.3), with corresponding Lie-isotopic brackets

$$[A \hat{,} B] = \frac{\partial A}{\partial x^{i_1}} 12^{i_1}{}_k(x)\ \Omega^{k i_2}(x) \frac{\partial B}{\partial x^{i_2}} \tag{9.89a}$$

$$\hat{1}_2 = \hat{T}_2{}^{-1}, \quad (\hat{\Omega}^{i_1 i_2}) = (\|\Omega_{k_1 k_2}\|^{-1}), \tag{9.89b}$$

where $\hat{1}_2 = \hat{T}_2{}^{-1}$ is the isounit of the universal enveloping associative algebra of the Lie-isotopic algebra with brackets (9.89) on $T^*\hat{M}_2(x,\hat{\Re})$. "Santilli's isosymplectic geometry" is the geometry of the symplectic-isotopic manifolds.

As an illustration, we shall now work-out an explicit model of isosymplectic-isotopic manifolds (*loc. cit.*). For physical applications it is sufficient to consider the *canonical isosymplectic- manifold*, i.e., the isomanifold of Definition 9.1 where Birkhoff's tensor Ω is replaced by the simpler canonical tensor ω.

Let us consider again the physical realization of the abstract $T^*\hat{M}_2(x,\hat{\Re})$ manifold as the cotangent bundle $T^*\hat{E}_2(r,\hat{\Re})$ with local coordinates

$$a = (a^\mu) = (r, p) = (r_i, p_i), \quad \mu = 1, 2, \ldots, 2n, \quad i = 1, 2, \ldots, n, \tag{9.90}$$

where r represents the Cartesian coordinates and p the linear mom-enta.

Then, we can introduce the *canonical one-isoform* on $T^*\hat{E}_1(r,\hat{\Re})$ of the particular type

$$\hat{\Phi}^\circ{}_1 = R^\circ{}_\mu \, T_1{}^\mu{}_\nu \, \hat{\partial} x^\nu, \tag{9.91a}$$

$$R^\circ = (p, 0), \tag{9.91b}$$

$$T_1 = \mathrm{diag.}\,(b^2{}_1, \ldots, b^2{}_{2n}) > 0, \quad b_k > 0 \tag{9.91c}$$

$$T_1{}^i{}_j = \delta^i{}_j \, b^2{}_j \quad \text{(no sum)}, \tag{9.91d}$$

Its isoexterior derivative on $T^*\hat{E}_2(r,\hat{\mathfrak{R}})$ is given by

$$\hat{\Phi}^\circ{}_2 = \hat{\partial}\hat{\Phi}^\circ{}_1 =$$

$$= \tfrac{1}{2}[\omega_{\mu_1\mu_2} b^2{}_{\mu_1} b^2{}_{\mu_2} + (R^\circ{}_{\mu_2} \frac{\partial b^2{}_{\mu_2}}{\partial a^{\mu_1}} - R^\circ{}_{\mu_1} \frac{\partial b^2{}_{\mu_1}}{\partial a^{\mu_2}})] \, \hat{\partial} a^{\mu_1} \hat{\wedge} \hat{\partial} a^{\mu_2} \tag{9.92}$$

and it always admits the factorization

$$\hat{\Phi}_2 = \hat{\partial}\, \hat{\Phi}_1 = \tfrac{1}{2}\, \omega_{\mu_1 \nu} \times \hat{T}_2{}^\nu{}_{\mu_2} \, \hat{\partial} a^{\mu_1} \hat{\wedge} \hat{\partial} a^{\mu_2}, \tag{9.93}$$

with

$$T_{2\nu}{}^{\mu_2} = b^{2\,\mu_2} b^2{}_\nu + \omega^{\mu_2 \rho} \, (R^\circ{}_\nu \frac{\partial b^2{}_\nu}{\partial a^\rho} - R^\circ{}_\rho \frac{\partial b^2{}_\rho}{\partial a^\nu}). \tag{9.94}$$

The isomanifold $T^*\hat{E}_2(r,\hat{\mathfrak{R}})$ equipped with two-isoform (9.93) is isosymplectic when T_2 coincides with its isotopic element.

Under these conditions, the generalized brackets characterized by structure (9.93)

$$[A \hat{,} B] = \frac{\partial A}{\partial a^{\mu_1}} \, \omega^{\mu_1 \nu} \times \hat{1}_{2\nu}{}^{\mu_2} \, \frac{\partial B}{\partial a^{\mu_2}}, \tag{9.95a}$$

$$\hat{1}_2 = \hat{T}_2{}^{-1}, \tag{9.95b}$$

are indeed Lie-Santilli and exhibit the isounit $\hat{1}_2$ of $T^*\hat{E}_2(r,\hat{\mathfrak{R}})$ directly in their structure, as desired.

A simple example is given by

$$T_2 = \text{diag.}\,(b^2{}_1, ..., b^2{}_{2n}) > 0, \quad b_k > 0, \tag{9.96}$$

The interested reader can work out an endless number of specific cases of isosymplectic-isotopic manifolds of both Birkhoff-Santilli and Hamilton-Santilli type. For specific examples, see Santilli (1991b).

We close this section with a comparative analysis of the isosymplectic geometry of this section and the Birkhoffian-Santilli mechanics. In Sect. 7 we outlined the isosymplectic two-forms, Definition 7.1; however, they were not symplectic *isoforms*. In fact, the one-form on $T^*\hat{E}_1(r,\Re)$ of this section

$$\hat{\Phi}_1 = R_\mu(a)\, T^\mu{}_\nu(a)\, \hat{d}x^\nu, \tag{9.97}$$

formally coincides with those of Sect. 7 in a fixed local chart in which $dx \equiv \hat{d}x$.

However, forms (9.97) are characterized in Sect. 7 by the ordinary calculus of differential forms. In fact, the main geometrical structure of Definition 7.1 is the *conventional exterior derivative of an exact conventional two-form*,

$$\Phi_2 = d(\Phi_1). \tag{9.98}$$

Since the Poincaré Lemma does indeed apply to the exact two-form $\hat{\Phi}_2$, we have

$$d\Phi_2 \equiv 0, \tag{9.99}$$

and the isotopy of Definition 7.1 then follows.

Santilli brought the notion of symplectic isotopy to its most general possible form, by introducing the isodifferential calculus of isoforms, with isoexterior derivatives $\hat{d}$, and then computed the two-isoforms

$$\hat{\Phi}_2 = \hat{d}(\hat{\Phi}_1). \tag{9.100}$$

The Poincaré-Santilli Lemma then ensures that

$$\hat{d}\hat{\Phi}_2 \equiv 0. \tag{9.101}$$

The infinite isotopies of Definition 9.1 then follows.

The direct applicability of the isosymplectic geometry for the

characterization of nonlinear, nonlocal and nonhamiltonian systems (1.1) then follows from the isoanalytic representations of Sect. 7.

Moreover, the isosymplectic geometry offered pragmatic means for the construction of a new generation of relativities. In fact, given Galilei's or Einstein's relativity for the description of a local, Hamiltonian, exterior dynamical system (Sect. 1) with canonical two-form ω and Hamiltonian H, one can construct an infinite number of covering, isotopic relativities for nonlinear, nonlocal and nonhamiltonian interior dynamical systems via the same Hamiltonian H and the tensorial product $\omega\times\hat{T}$ subject to the condition of remaining a nowhere degenerate, exact, symplectic two-form (see Santilli (1991b) for details).

I.10: ISOAFFINE GEOMETRY.

I shall now review *Santilli's (1988d), (1991b) affine-isotopic geometry* or *isoaffine geometry* for short. The objective is that of achieving a generalization of the current local-differential character of the affine geometry into a nonlocal-integral form capable of treating systems of type (1.1), and identify the expected, consequential generalization of the notions of curvature, parallel transport, geodesic, etc.

The literature in the conventional affine geometry is predictably vast. Among the earliest references, the presentation by Schrödinger (1950) still has considerable value. In this section we shall continue to follow the treatise by Lovelock and Rund (1975) of which Santilli preserves the notation mostly unchanged for clarity in the comparison of the results.

To our best knowledge, the isotopies of the affine geometry have been investigated for the first time in Santilli (1988d), developed in more details in (1991b). Their application to the isotopies of Einstein's gravitation appeared in Santilli (1988d) and (1991d).

The understanding of this and of the following sections requires a prior knowledge of the following notions introduced in preceding sections: *isofields* $\hat{\mathfrak{R}}$, *isovector spaces* $\hat{V}$ and *isometric spaces* $\hat{M}$, with particular reference to the *isoeuclidean space* $\hat{E}(r,\hat{\delta},\hat{\mathfrak{R}})$ and the *isominkowski space* $\hat{M}(x,\hat{\eta},\hat{\mathfrak{R}})$.

The implications of isotopies for differentiable manifolds were identified in the preceding section via the notion of *isodifferentials* $\hat{d}x$ and related *isoexterior calculus*.

Santilli continued his studies by identifying the implications of isodifferentials for the notions of connections, curvature, etc.

Let $M(x,\mathfrak{R})$ be an n-dimensional *affine space* (Lovelock and Rund (*loc. cit.*))

here referred as a differentiable manifold with local coordinates $x = (x^i)$, $i = 1, 2, ..., n$, over the reals $\Re$. We shall denote: the conventional scalars on $M(x,\Re)$ with $\phi(x)$; contravariant and covariant vectors with $X^j(x)$ and $X_j(x)$, respectively; and mixed tensors of rank (r,s)

$$X^{(r,s)} = X^{j_1 j_2 ... j_r}{}_{k_1 k_2 ... k_s}(x). \tag{10.1}$$

Unless otherwise stated, all tensors considered on $M(x,\Re)$ will be assumed hereon to be local-differential and to verify all needed continuity conditions.

DEFINITION 10.1: The infinite class of isotopic liftings $\hat{M}(x,\hat{\Re})$ of an affine space $M(x,\Re)$, called "Santilli's isoaffine spaces", are characterized by the same local coordinates x and the same local-differential tensors $X^{(r,s)}$ of $M(x,\Re)$ but now defined with respect to the isotopic liftings of the field

$$M(x,\Re) \Rightarrow \hat{M}(x,\hat{\Re}) : \hat{\Re} = \Re\hat{1}, \tag{10.2}$$

for all infinitely possible isounits $\hat{1}$ in n×n dimension which are nowhere singular and Hermitean, but otherwise possess an arbitrary, generally nonlinear and nonlocal dependence on the variables x, their derivatives with respect to an independent parameter s of arbitrary order, and any other quantity needed for physical applications, such as density μ of the interior physical medium considered, its temperature τ, its index of refraction n, etc.

$$\hat{1} = \hat{1}(x, \dot{x}, \ddot{x}, \mu, \tau, n, ...). \tag{10.3}$$

In this and in the next two sections we shall study isoaffine spaces for arbitrary isounits $\hat{1}$. Nevertheless, it may be recommendable to keep in mind Santilli's intended use of the theory, that of attempting a more general formulation of the *interior* gravitational problem, which is capable of recovering identically the conventional gravitational theories for the exterior problem (see Santilli (1988d, (1991d) for details).

As a result, the reader should keep in mind that:

1) The isounits $\hat{1}$ are intended to be different than the trivial unit I only in a well identified region of space, generally given by the interior of the minimal surface $S°$ encompassing all matter of the body considered, including its boundary (e.g., the interior of Jupiter);

2) The isounits $\hat{1}$ shall represent the nonlinear, nonlocal and nonlagrangian[16]

forces of the interior gravitational problem, as well as the generally inhomogeneous and anisotropic character of interior physical media; and

3) All possible isounits $\hat{1}$ shall recover the trivial units I in the exterior of the surface S°, so as to permit the recovering in their entirety of the conventional, exterior gravitational theories.

Thus, to avoid major misrepresentations of Santilli's discoveries, the reader should keep in mind throughout our analysis that the generalized geometries apply only within physical media, while recovering the conventional geometries in empty space by construction.

As a matter of fact, the transition from motion in a curved empty space, to a curved space filled up with a physical medium is precisely representable with the transition from conventional to Santilli's geometries.

As done in the preceding sections, the isounit $\hat{1}$ will be assumed to be nonsingular, real-valued and symmetric, $\hat{1}^\dagger \equiv \hat{1}$, $\hat{1} = (\hat{1}^i{}_j) = (\hat{1}_j{}^i)$, $\hat{1} \neq 0$.

The *isotopic element* $T = T(x, \dot{x}, ...)$ of the theory can then be written

$$\hat{1} = T^{-1}, \quad T = (T_i{}^j) = (T^j{}_i). \tag{10.5}$$

A first salient feature of the liftings $M(x,\Re) \Rightarrow \hat{M}(x,\hat{\Re})$ is that the conventional *linear and local transformations*, i.e., the linear, right, modular, associative transformations on $M(x,\Re)$

$$x' = A\,x, \tag{10.6}$$

must now be necessarily generalized into the *Santilli's isolinear and isolocal transformations* on $\hat{M}(x,\hat{\Re})$, i.e., the *right, modular, isoassociative transformations* studied in Sect. 4,

$$\bar{x} = A * x \overset{\text{def}}{=} ATx, \tag{10.7}$$

where T is fixed.

In turn, the lifting $Ax \Rightarrow A{*}x$ has a number of consequences. First, it permits the treatment of nonlocal-integral structures which would be otherwise precluded by the conventional theory of affine spaces.

This is readily done via the embedding of all nonlocal-integral terms in the isotopic element of the theory. The insensitivity of the affine geometry to the topology of its unit then ensures the achievement of a mathematically consistent structure.

Secondly, isotransformations (10.7) are called *isolinear* and *isolocal* (Sect. 4)

[16] See Footnote 3 of Sect. 6.

in the sense that they verify all abstract linearity and locality conditions on $\hat{M}(x,\hat{\mathfrak{R}})$. Nevertheless, *they are generally nonlinear and nonlocal when written in the original space* $M(x,\mathfrak{R})$, i.e.

$$\bar{x} = A * x = A\, T(x, \dot{x}, \ddot{x}, ..)\, x \tag{10.8}$$

Santilli's liftings $Ax \Rightarrow A{*}x$ imply that all conventional contractions of indices are now lifted via the insertion of the isotopic element, i..e.,

$$A^i{}_j x^j \quad \Rightarrow \quad A^i{}_j T^j{}_k x^k. \tag{10.9}$$

Let us also recall that the use of conventional transformations (10.6) on the isotopic spaces $\hat{M}(x,\hat{\mathfrak{R}})$ would violate the condition of (iso) linearity. This illustrates the *necessity* of the liftings $Ax \Rightarrow A{*}x$.

Finally, we assume the reader is familiar with the fact that all distinctions between conventional transformations (10.6) and their isotopic forms (10.7) cease to exist, by construction, at the abstract, realization-free level. Thus, by their very conception, *isoaffine spaces are a more general realization of the mathematical axioms of the conventional spaces.*

This ultimate geometric equivalence ensures the mathematical consistency of the liftings. As a matter of fact, the equivalence can be used to verify the consistencies of conventional treatments, as we shall see.

Despite this axiomatic equivalence, the differences between the affine and the isoaffine geometries are rather deep.

Recall in the conventional case that, given two contravariant vectors x_1 and x_2 on $M(x,\mathfrak{R})$, their difference Δx is a contravariant vector iff the transformation is *linear* (as well as local).Similarly, Δx is a contravariant vector on $\hat{M}(x,\hat{\mathfrak{R}})$ iff the transformation is *isolinear* (as well as isolocal). The following result then holds(see also Propositions 3.1).

PROPOSITION 10.1 (Santilli (1988d), (1991b)): For any given (sufficiently smooth) nonlinear and/or nonlocal transformation on $M(x,\mathfrak{R})$, there always exists an isounit $\hat{1}$ under which the transformation becomes isolinear and/or isolocal, respectively, on $\hat{M}(x,\hat{\mathfrak{R}})$, $\hat{\mathfrak{R}} = \mathfrak{R}\hat{1}$. Similarly, for any given coordinate differences Δx of two contravariant vectors on $M(x,\mathfrak{R})$ which does not transform contravariantly, there always exists an isotope $\hat{M}(x,\hat{\mathfrak{R}})$ of $M(x,\mathfrak{R})$ under which Δx transforms isocontravariantly.

The *left and right modular isotransformations* are evidently defined by

$$\bar{x}^t = x^t * A^t = x^t\, T\, A^t, \tag{10.10a}$$

$$\bar{x} = A*X = ATx, \tag{10.10b}$$

where t denotes conventional transpose. The inverse, right-modular transformations are given by the isotopic rule

$$x = A^{\hat{-1}} * \bar{x} = A^{\hat{-1}}\, T\, x, \tag{10.11}$$

where $A^{\hat{-1}}$ is the *isoinverse*, i.e., it verifies the isotopic rules

$$A^{\hat{-1}} * A = A * A^{\hat{-1}} = \hat{1}, \tag{10.12}$$

and, when considering the isotopy in the new coordinate system, Santilli puts

$$T = \bar{T}(\bar{x}, \dot{\bar{x}}, ...) \equiv T(x, \dot{x}, ...). \tag{10.13}$$

Note the preservation of the isotopic element for the left and inverse isotransformations. This preservation is ensured by the assumed Hermiticity of the element T and it is at the very foundations of the Lie-Santilli theory reviewed in Sect. 6,[17]

$$[A \hat{,} B] = A*B - B*A = ATB - BTA. \tag{10.14}$$

$\hat{M}(x,\hat{\mathfrak{R}})$ is then the correct *isomodule* for the isorepresentations of Lie-Santilli algebras characterized by product (10.13) (App. D).

If the Hermiticity of T is relaxed, the right isotopic element becomes different than the left one

$$\bar{x}' = A > x \stackrel{\text{def}}{=} A\, T\, x, \qquad \bar{x}^t = x^t < A^t \stackrel{\text{def}}{=} x^t T^\dagger A^t, \tag{10.15a}$$

$$T \neq T^\dagger, \tag{10.15b}$$

This signals the necessary emergence of the covering *Lie-admissible theory* (Santilli (1967), (1968), (1978a) (1981a)) with basic product[18]

[17] Note the direct dependence of the *Lie* character of the theory from the *Hermiticity* of the isotopic element T.

[18] We here want to stress for the noninitiated reader that the *Lie-admissible* character of

$$(A\,,B) \;=\; A < B \;-\; B > A \;=\; AT^{\dagger}B \;-\; BTA, \tag{10.16}$$

verifying the axioms of the covering Lie-admissible algebra (Sect. 5). In this case the generalized affine space is the correct *isobimodule* of the Lie-admissible algebra (App. D).

Note that in this case Santilli has *two different isounits,*

$$1^{>} = T^{-1}, \quad {}^{<}1 = T^{\dagger -1}, \tag{10.17}$$

and two different isofields

$$\Re^{>} = \Re 1^{>}\;,\; {}^{<}\Re = {}^{<}1\,\Re, \tag{10.18}$$

or, equivalently, one single quantity ${}^{<}\Re^{>}$, representing both the right- or left-modular-isotopic action depending on the assumed conjugation in the isofield.

We shall reserve the name of *Santilli's affine-admissible spaces* (or *genoaffine* [19] *spaces*) and the symbol ${}^{<}M^{>}(x, {}^{<}\Re^{>})$ to the emerging structure.

Santilli's isoaffine spaces are conceived for the study of interior gravitation as a whole, i.e., in closed-isolated conditions. In fact, the antisymmetry of the Lie-Santilli product (10.14) ensures the conservation of the total energy,

$$dH/dt \;=\; [H\,\hat{,}\,H] \;=\; HTH \;-\; HTH \;\equiv 0, \tag{10.19}$$

and similar conservation follow for the other total quantities under a generalized internal structure evidently represented by the isotopic element T. As a result, isospaces $\hat{M}(x,\hat{R})$ are the fundamental ones of the analysis of the main text of this volume.

Santilli's genoaffine spaces instead imply the necessary study of gravitation in open-nonconservative conditions. In fact, owing to the lack of antisymmetry of the Lie-admissible product (10.16), we now have time-rate-of-variations of the energy H of the considered interior particle

$$dH/dt \;=\; (H\,,H) \;\neq\; 0, \tag{10.20}$$

while the remaining system is considered to be external. The affine-admissible spaces therefore are the fundamental ones on the still more general, Lie-admissible

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the emerging theory essentially depends on the *nonhermiticity* of the isotopic element T.

[19] See later on Figure 1 for the origin of this name.

approach, which is briefly indicated in the appendices without detailed treatment.

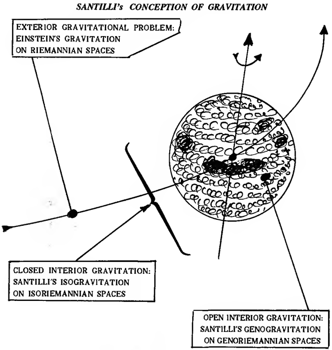

FIGURE 1: A schematic view of Santilli's conception of gravitation. The birth of all theories of relativities for the exterior problem can be identified with the first visual observation of the *Jovian system* by Galileo Galilei in 1906. Santilli stresses in several of his writings (1978a), (1981a), (1982a), (1988d), (1991d) that the birth of the new generation. of relativities for the interior problem can be identified with the visual observation, this time, of *Jupiter's structure*, as offered by contemporary telescopes or by the recent NASA missions. Such a visual observation reveals the following incontrovertible physical evidence:

a) Jupiter is a stable system when considered as a whole and, thus, it verifies conventional total conservation laws;

b) Jupiter's interior dynamics is essentially nonconservative, as established by vortices with continuously varying angular momenta.

c) In Jupiter's structure we therefore have internal exchanges of energy and other quantities, but always in such a way that they balance each other resulting in total conservation laws.

d) In a way parallel to the above, the orbit of each constituent (say, molecular) of Jupiter is unstable, but in such a way that the center-of-mass trajectory of the system is in a stable orbit.

e) Jupiter's interior dynamics is intrinsically irreversible, but, again, in such a way that the center-of-mass trajectory verifies in full the time reversal symmetry.

In order to achieve a quantitative representation of the above physical evidence, Santilli (1978b) introduced the notion of *closed nonhamiltonian systems*, that is, systems which verify total conventional conservation laws, nevertheless they possess nonlinear, nonlocal, and nonhamiltonian internal forces as in systems (1.1). He then worked out a number of theorems (Santilli (1985c)), today called *"Santilli's No Reduction Theorems"* which prove the impossibility of reducing Jupiter to an ideal collection of conservative constituents in stable orbits. This established the need beyond any credible scientific doubt of a suitable generalization of Einstein's gravitation for the interior problem which is capable of directly representing the above features a)–e) without any idealistic reduction to hypothetical conditions.

To state explicitly this basic point, *the insistence in the exact validity of Einstein's gravitation in the interior problem directly implies the acceptance of the perpetual motion within a physical environment* . In fact, the application of Einstein's gravitation for an interior test particle, say, a spaceship penetrating Jupiter's atmosphere, necessarily implies that the spaceship orbits inside Jupiter's atmosphere with a conserved angular momentum, trivially, from the necessary locally-Lorentz character of Einstein's gravitation, with consequently necessary local validity of the rotational symmetry and related conservation of the angular momentum.

Santilli then entered into a comprehensive study of closed nonhamiltonian systems at large, and of Jupiter's structure in particular, at the classical Galilean level, by achieving his *isogalilean relativity* (Santilli (1982a), (1988a), (1991d)), at the relativistic level, by achieving his *isospecial relativity* (Santilli (1983a), (1988c), (1991d)), and at the gravitational level by achieving his *isogeneral relativity* (Santilli (1988d), (1991d)).

To understand the dimension and inter-relations of the studies, the reader should be aware that Santilli (*loc. cit.*): proved the mutual compatibility of the isogalilean, isospecial and isogeneral relativities; he duplicated his study into two classes, one of isotopic and a second one of genotopic character (see below in this figure); and he studied this dual isogalilean-isospecial-isogeneral theories at both classical and operator levels.

The reader can now understand the difficulties by this author in presenting such a rather vast, diversified and interrelated scientific edifice encompassing a dual generalization of contemporary classical and quantum theoretical physics. In order to avoid major

confusions and misrepresentations, this author first decided to avoid any overview in the Foreword (which would be incomprehensible to the noninitiated reader beginning with the needed terminology); to restrict the review of this volume only to classical formulations; and delay the operator formulations to a possible, future, second volume.

Secondly, this author decided to limit the presentation up this section to Santilli's isotopies. This is motivated by the desire to bring the reader progressively into the complexities of Santilli's mathematical and physical conceptions, as well as by the fact fact that all advances reviewed until now, including the isosymplectic geometry, have an interest per se, independently from physical applications.

Moreover, and again contrary to general practices, this author decided to present first Santilli's conception of interior gravitation and then their isolorentzian and, in our sequence, isogalilean particularizations. This is due to the need, in the view of this Author, to understand first Santilli's notion of isoparallel transport and isogeodesic, in order to really understand his isospecial and isogalilean relativities. In fact, lacking such a prior knowledge, the reader is not in a position to understand, say, the preservation of the geodesic character in the transition from Galilei's historical uniform motion in vacuum, to Santilli's motion of the same body within a physical medium.

Finally, again for the purpose of minimizing possible confusions, this author elected to present the discoveries of isotopic type in the main text of this volume, and those of genotopic type in the appendices.

The reason why the mention of the genotopic theories is mandatory at this point is due to the fact that, as stressed by Santilli, no genuine advancement over Einstein's gravitation is possible without a dual representation, one side, of the global stability and reversibility of the center-of-mass trajectory and, on the other side, of the nonlocal-nonhamiltonian internal forces, as well as of the interior irreversibility, as majestically established by Jupiter.

To understand the second family of generalizations, let us recall that, in his historical memoir of 1978a, Santilli introduced two fundamental notions, each one with vast implications for mathematics and physics:

I) *ISOTOPIC MAPPINGS*, which are the *"axiom preserving mappings"* studied in the main text of this volume. The prefix *"iso"* in *isosymplectic, isoaffine and isoriemannian* geometries, therefore, stands to indicate the *preservation* of the abstract axioms of the original geometries, as now well known. The physically and mathematically most important isotopy is that of a conventional Lie algebra we have studied in the main text

$$AB - BA \Rightarrow \text{ISOTOPY} \Rightarrow ATB - BTA, \quad T^\dagger = T.$$

II) *GENOTOPIC MAPPINGS*, which were conceived by Santilli as *"axioms inducing mappings"*. Therefore, the prefix *"geno"* in *genosymplectic, genoaffine and genoriemannian* geometries stands to indicate the *alteration* of the original axioms by conception in favor of novel structures. More specifically, given a (generally nonassociative) algebra U with elements a, b, c, ... and abstract product ab over a field F which verifies a given set of axioms A (say, those of a Lie algebra, of a Jordan algebra, etc.),

then Santilli (1978a) introduced, apparently for the first time, the notion of *genetope* U^X of U as the same vector space U, but equipped with a new product, say, a×b which violates the original axioms A and verifies instead a different set of axioms A'. In this sense, Santilli "induces" new axioms from the original ones. The physically and mathematically most important genotopy is that of a Lie algebra into a Lie-admissible algebra treated in this App. A

$$AB - BA \Rightarrow \text{GENOTOPY} \Rightarrow AT^{\dagger}B - BTA, \quad T^{\dagger} \neq T.$$

As indicated in the text, the isotopy preserves the Lie character of the original formulation, and therefore its emphasis is on total conservation laws. By contrast, the genotopy alters the original Lie character, and it is requested when considering the more general open nonconservative systems. In fact, as we shall outline in Appendix A, Santilli's Lie-admissible formulations characterize the time-rate-of-variation of physical quantities.

We are now sufficiently equipped to begin the presentation of *Santilli's (1988d), (1991d) dual generalization-covering of Einstein's gravitation,* called *Santilli's isogravitation* and *genogravitation*, according to the following main lines:

A) CLOSED INTERIOR GRAVITATIONAL PROBLEMS, which are studied via the infinite number of possible isoaffine and isoriemannian geometries characterized by the infinitely possible different interior physical media for each given gravitational mass represented by the infinitely possible isotopic elements $T = T^{\dagger}$ for each given metric $g_{\mu\nu}$. In this case, the emphasis is in the *CONSERVATION LAWS* for total quantities, the *STABILITY* of the system as a whole and of its center-of-mass trajectory, and the *REVERSIBILITY* of its center-of-mass trajectory in vacuum, all this while permitting local internal *NONLOCAL* and *NONCONSERVATIVE*, as well as *NONHAMILTONIAN* forces.

B) OPEN INTERIOR GRAVITATIONAL PROBLEMS, which are studied via the infinitely possible genoaffine and genoriemannian geometries with genotopic elements $T \neq T^{\dagger}$, and with different actions to the right and to the left (forward and backward in time). A typical case is the description of a test particle while moving within the physical medium of the interior gravitational problem, such as a spaceship penetrating within Jupiter's atmosphere. In this case, the primary emphasis is in the representation of the *NONCONSERVATIVE* character of the test particle, the *INSTABILITY* of its orbit, and the *IRREVERSIBILITY* of the process to avoid excessive approximation of physical reality.

C) EXTERIOR GRAVITATIONAL PROBLEMS IN VACUUM, which coincides with the conventional Einstein's gravitation by central assumption of Santilli's theories. In fact, Santilli's isogravitation and genogravitation are restricted by the central condition that their isotopic and genotopic element T reduces to the identity when motion occurs in vacuum, i.e.,

$$T_{|r > S^{\circ}} = I$$

(where S° is the surface encompassing all matter of the body considered, including its possible atmosphere). Under the above condition, both the isoriemannian and the

genoriemannian geometries recover the conventional Riemannian geometry, with the consequential validity of the conventional Einstein's gravitation.

Santilli's isospecial relativity will then naturally emerge in the local tangent planes, and his isogalilean formulation can be easily derived via the techniques of isogroup contractions. As indicated earlier, in the remaining sections of this volume we shall outline the isotopic formulations, while the more general genotopic formulations will be presented in the appendices.

The understanding is that all isotopic formulations are a particular case of the genotopic ones at all levels of study, whether classical or quantum mechanical. This illustrates the reason why, quite appropriately, the Estonian Academy of Science (1989) of Tartu recently honored Santilli by including his name in a chart identifying some of the most famous contributors in mathematical physics from 1800 to today, *because of his studies in Lie-admissible algebras*, and with the paper (Santilli (1967)) which signals precisely the birth of the Lie-admissible formulations in physics.

A final aspect should be indicated for the receptive reader. As well known, Albert Einstein found all the needed mathematics ready for the construction of both, his special and general relativities. In fact, for the construction of the special relativity he found available the fundamental Lie's theory, the Lorentz and Poincare' symmetries, and other mathematical tools. Similarly, for the construction for his theory of gravitation he found available the Riemannian geometry.

By contrast, a most remarkable achievement by Ruggero Maria Santilli, a theoretical physicist, is that he had to construct, first, the novel mathematical tools needed for his dual treatment of nonlinear, nonlocal and nonhamiltonian systems and, then, construct his generalized Galilean, special and general relativities.

We now study the *affine-isotopic geometry*, or *isoaffine geometry*, i.e., the isotopic liftings of the conventional geometry characterized by isotransformations (10.8).

Recall from Sect. 9 that, in the conventional case, the differentials of the two coordinates x and x′ are given by the familiar forms dx′ and dx with interconnecting rule

$$dx' = A\,dx, \qquad dx'^{\,i} = A^i{}_j\,dx^j. \tag{10.21}$$

But the same interconnection does not hold for the differentials $d\bar{x}$ and dx because of property (10.7), i.e., by central assumption of isotopy, $d\bar{x} \neq A\,dx$.

Following Sect. 9, Santilli therefore introduces the generalized notion of *isodifferentials* $\hat{d}\bar{x}$ and $\hat{d}x$ when interconnected by the isotopic law

$$\hat{d}\bar{x} = A * \hat{d}x, \qquad \hat{d}\bar{x}^i = A^i{}_j\,T^j{}_k\,\hat{d}x^k. \tag{10.22}$$

Similarly, recall from Sect. 9 the *isodifferential of an isoscalar* $\phi(x)$ on

$\hat{M}(x,\hat{\Re})$

$$\hat{\partial}\phi(x) = \frac{\partial\phi}{\partial x} * \hat{\partial}x = \frac{\partial\phi}{\partial x^i} T^i{}_j \hat{\partial}x^j \tag{10.23}$$

where the partial derivative is the conventional one, as well as the *isodifferential of a contravariant isovector* $X = (X^i(x))$ on $\hat{M}(x,\hat{\Re})$

$$\hat{\partial}X = \frac{\partial X}{\partial x} * \hat{\partial}x, \qquad \hat{\partial}X^i = \frac{\partial X^i}{\partial x^j} T^j{}_k \hat{\partial}x^k, \tag{10.24}$$

The above quantity then imply the *isotransformation laws* of the *contravariant isovector*

$$\bar{X}(\bar{x}) = \frac{\partial\bar{x}}{\partial x} * X(x), \qquad \bar{X}^i = \frac{\partial\bar{x}^i}{\partial x^j} T^j{}_k X^k(x), \tag{10.25}$$

Recall also that, while in the conventional (linear) case $x' = Ax$, $\partial x'/\partial x = A$, we now have on $\hat{M}(x,\hat{\Re})$

$$\frac{\partial\bar{x}^i}{\partial x^j} = A^i{}_k T^k{}_j + A^i{}_k \frac{\partial T^k{}_r}{\partial x^j} x^r, \tag{10.26)}$$

Similarly, Santilli has the *isotransformations* of a *contravariant isotensor* X^{ij} of rank two on $\hat{M}(x,\hat{\Re})$

$$\bar{X}^{(2)}(\bar{x}) = \frac{\partial\bar{x}}{\partial x} * \frac{\partial\bar{x}}{\partial x} * X^{(2)}(x), \qquad \bar{X}^{ij}(\bar{x}) = \frac{\partial\bar{x}^i}{\partial x^r} T^r{}_p \frac{\partial\bar{x}^j}{\partial x^s} T^s{}_q X^{pq}(x), \tag{10.27}$$

with similar extension to higher orders, as well as contravariant isotensors of rank (0.s) and generic tensors of rank (r.s).

The reader should also recall from Sect. 9 the identity of the above isoquantities with the conventional quantities.

The following derivatives the *isodifferential of a contravariant isovector-field*

$$\hat{\partial}\bar{X}^j = \frac{\partial\bar{X}^j}{\partial \bar{x}^k} T^k{}_r \hat{\partial}\bar{x}^r =$$

$$\partial \overline{x}^k$$

$$= \frac{\partial^2 \overline{x}^j}{\partial x^s \partial x^i} T^i_{\ r} X^r \partial x^s + \frac{\partial \overline{x}^j}{\partial x^i} T^i_{\ r} \frac{\partial X^r}{\partial x^s} \partial x^s + \frac{\partial \overline{x}^j}{\partial x^i} \frac{\partial T^i_{\ r}}{\partial x^s} X^r \partial x^s. \qquad (10.28)$$

Santilli (*loc. cit.*) then introduces the *isocovariant (or isoabsolute) differential* $\hat{D}X^r$

$$\hat{D}X^j = \partial X^j + \hat{P}^j(x, X, \partial x), \qquad (10.29)$$

under the condition that it preserves the original axioms (Lovelock and Rund (*loc.cit.*), p. 68), i.e.,

1) $\hat{D}(X^j + Y^j) = \hat{D}X^j + \hat{D}Y^j$, which can hold iff $\hat{P}^j$ is isolinear in X^r;

2) $\hat{D}X^j$ is isolinear in ∂x^s; and

3) $\hat{D}X_j$ transforms as a contravariant isovector.

By again using Lovelock-Rund's symbols with a "hat' to denote isotopy, we can write

$$\hat{D}X^j = \partial X^j + \hat{\Gamma}_{h\ k}^{\ j} T^h_{\ r} X^r T^k_{\ s} \partial x^s, \qquad (10.30)$$

where the $\hat{\Gamma}$'s were called the component of an *isoaffine connection.*

By lifting the conventional procedure, one can readily see that the necessary and sufficient conditions for the n^3 quantities $\hat{\Gamma}_{m\ n}^{\ s}$ to be the coefficient of an isoaffine connection are given by

$$\hat{\Gamma}_{m\ p}^{\ j} T^m_{\ r} \frac{\partial \overline{x}^r}{\partial x^s} T^s_{\ t} X^t T^p_{\ q} \frac{\partial \overline{x}^q}{\partial x^w} T^w_{\ z} \partial x^z =$$

$$= \frac{\partial \overline{x}^j}{\partial x^r} T^r_{\ s} \hat{\Gamma}_{m\ n}^{\ s} T^m_{\ p} X^p T^n_{\ q} \partial x^q - \frac{\partial^2 \overline{x}^j}{\partial x^s \partial x^i} T i^r X^r \partial x^s +$$

$$+ \frac{\partial x^j}{\partial x^i} T^h_{\ r} \frac{\partial X^r}{\partial x^s} (T^s_{\ t} \partial x^t - \partial x^s) - \frac{\partial \overline{x}^j}{\partial x^i} \frac{\partial T^i}{\partial x^s} \hat{X}^r \partial x^s. \qquad (10.31)$$

As in the conventional case, the $\hat{\Gamma}$'s do not constitute a tensor of rank (1.2). The extra terms in conditions (10.31), therefore, do not affect the consistency of the isoaffine geometry, but constitute the *desired generalization.*

An important particular case occurs when T is a constant, which is the case

when the characteristic isotopic functions representing the interior physical medium are subjected to a suitable average into constants (see next chapter). In this case the isotopy of the conventional terms persists, but the additional terms are null. Finally, note that all conventional notions and properties are admitted as a trivial particular case by the isoaffine geometry whenever T = 1.

The extension of the above results to the *isocontravariant derivatives* is evidently given by

$$\hat{D}X_j = \hat{\partial}X_j - \hat{\Gamma}_j{}^s{}_n T_s{}^r X_r T^n{}_p \hat{d}x^p . \tag{10.32}$$

As a result, *the isocovariant derivative of a scalar coincides with the isodifferential*, as in the conventional case, i.e.,

$$\hat{D}\phi = \hat{D}(X^i X_i) = \hat{\partial}\phi . \tag{10.33}$$

The isoaffine connection is *symmetric* if the following property is verified

$$\hat{\Gamma}_m{}^s{}_n = \hat{\Gamma}_n{}^s{}_m . \tag{10.34}$$

The following property can be easily proved (but carries important physical consequences).

PROPOSITION 10.2 (Santilli (loc. cit.)): The isotopic image $\hat{\Gamma}_h{}^j{}_k$ of a conventional, symmetric, affine connection $\Gamma_h{}^j{}_k = \Gamma_k{}^j{}_h$ is not necessarily symmetry.

The isotopic liftings of all remaining properties of covariant derivatives, as well as the extension to the isocovariant differential of tensors, will be left for brevity to the interested reader.

It is easy to see that the isocovariant (isoabsolute) differential preserves the basic axioms of the conventional differential, i.e., (Lovelock and Rund ((loc. cit.), p.74):

AXIOM 1: The isocovariant differential of a constant is identically null; that of a scalar coincides with the isodifferential; and that of a tensor of rank (r,s) is a tensor of the same rank.
AXIOM 2: The isocovariant differential of the sum of two tensors of the same rank is the sum of the isoabsolute differentials of the individual tensors. And
AXIOM 3: The isocovariant differential of the product of two tensors of the same

rank verifies the conventional chain rule of differentiation.

By following again the pattern of the conventional formulation, and as a natural generalization of the isocovariant differential, Santilli introduces the *isocovariant derivative* of a contravariant vector field X^p

$$X^j{}_{\uparrow k} \stackrel{\text{def}}{=} \frac{\partial X^j}{\hat{\partial} x^k} + \hat{\Gamma}^{\,j}_{h\,k}\, T^h{}_r\, X^r, \tag{10.35}$$

under which the isocovariant differential can be written

$$\hat{D}X^j = X^j{}_{\uparrow k}\, T^k{}_s\, \hat{\partial} x^s. \tag{10.36}$$

It is an instructive exercise for the interested reader to prove that the isocovariant derivatives (10.35) constitute the components of a (1.1) isotensor.

It is also easy to verify that the isocovariant derivatives preserve the axioms of the conventional covariant derivatives (Lovelock and Rund (*loc. cit.*), p. 77):

AXIOM 1': The isocovariant derivative of a constant is identically null; that of a scalar is equal to the conventional partial derivative; and that of an isotensor of rank (r. s) is an isotensor of rank (r. s+1).
AXIOM 2': The isocovariant derivative of the sum of two tensors of the same rank is the sum of the isocovariant derivatives of the individual tensors. And
AXIOM 3': The isocovariant derivative of the product of two isotensors of the same rank is that of the usual chain rule of partial derivatives.

Axioms 1, 2, 3 and 1', 2', 3' imply the most important result of this section, which can be expressed via the following

PROPOSITION 10.3 (Santilli (loc. cit.)): Under sufficient continuity conditions, all infinitely possible isotopic liftings of an affine geometry coincide with the same geometry at the abstract, coordinate-free level.

In actuality, the capability of our isotopies of preserving the basic axioms is such that, the isotopic liftings can be used as a test of geometric consistency of a *conventional* theory.

In fact, *if a given property is not preserved under isotopy, the definition of the property itself is geometrically incomplete* . As we shall see in the next

section, this is precisely the case of the historical Einstein's tensor.

We now pass to the review of a central notion of the isoaffine geometry, the generalized curvature, called by Santilli *isocurvature*, and the generalized torsion, called *isotorsion*, which are inherent in the isoaffine geometry prior to any introduction of an isometric (to be done in the next section).

For this purpose, let us study the lack of commutativity of the isocovariant derivatives on isoaffine spaces $\hat{M}(x,\hat{\Re})$ with respect to an arbitrary, not necessarily symmetric, isoconnection $\hat{\Gamma}_h{}^j{}_k$. Via a simple isotopy of the corresponding equations (see Lovelock–Rund (*loc. cit.*), pp 82–83), and by noting that

$$X^j{}_{\uparrow h \uparrow k} = \frac{\partial}{\partial x^k}(X^j{}_{\uparrow h}) + \hat{\Gamma}^2{}_p{}^j{}_k\, T^p{}_q\,(X^q{}_{\uparrow h}) - \hat{\Gamma}^2{}_h{}^p{}_k\, T_p{}^q (X^j{}_{\uparrow q}), \qquad (10.37)$$

Santilli gets the expression

$$X^j{}_{\uparrow h \uparrow k} - X^j{}_{\uparrow k \uparrow h} = \Big(\frac{\partial \hat{\Gamma}^2{}_l{}^j{}_h}{\partial x^k} - \frac{\partial \hat{\Gamma}^2{}_l{}^j{}_k}{\partial x^h} +$$

$$+ (\hat{\Gamma}^2{}_m{}^l{}_k\, T^m{}_r\, \hat{\Gamma}^2{}_l{}^r{}_h - \hat{\Gamma}^2{}_m{}^l{}_h\, T^m{}_r\, \hat{\Gamma}^2{}_l{}^r{}_k)\, T^l{}_s\, X^s -$$

$$- (\hat{\Gamma}^2{}_h{}^l{}_k - \hat{\Gamma}^2{}_k{}^l{}_h) T_l{}^r X^j{}_{\uparrow r} - (\hat{\Gamma}^2{}_l{}^j{}_h \frac{\partial}{\partial x^k} - \hat{\Gamma}^2{}_l{}^j{}_k \frac{\partial}{\partial x^h})\, X^r, \qquad (10.38)$$

DEFINITION 10.2 (Santilli (loc. cit.)): The "isocurvature tensor" of a vector field X_r on an n-dimensional isoaffine space $\hat{M}(x,\hat{\Re})$ is given by the isotensor of rank (1.3)

$$\hat{R}_l{}^j{}_{hk} = \frac{\partial \hat{\Gamma}_l{}^j{}_h}{\partial x^k} - \frac{\partial \hat{\Gamma}_l{}^j{}_k}{\partial x^h} +$$

$$+ \hat{\Gamma}_m{}^j{}_k\, T^m{}_r\, \hat{\Gamma}^2{}_l{}^r{}_h - \hat{\Gamma}_m{}^j{}_h\, T^m{}_r\, \hat{\Gamma}^2{}_l{}^r{}_k +$$

$$+ \hat{\Gamma}_r{}^j{}_h \frac{\partial T^r{}_s}{\partial x^k} \hat{1}^s{}_l - \hat{\Gamma}_r{}^j{}_k \frac{\partial T^r{}_s}{\partial x^h} \hat{1}^s{}_l\,; \qquad (10.39)$$

while the "isotorsion tensor" is given by

$$\hat{\tau}_h{}^l{}_k = \hat{\Gamma}_h{}^l{}_k - \hat{\Gamma}_k{}^l{}_h\,; \qquad (10.40)$$

Expression (10.38) can then be written

$$X^{j}{}_{\uparrow h \uparrow k} - X^{j}{}_{\uparrow k \uparrow h} = \hat{R}_{l}{}^{j}{}_{hk} T^{l}{}_{s} X^{s} - \hat{\tau}_{h}{}^{l}{}_{k} T^{s}{}_{l} X^{j}{}_{\uparrow s}. \tag{10.41}$$

Comparison with the corresponding conventional expression (Eq.s (6.9), p. 83, Lovelock and Rund (*loc.cit.*)) is instructive to understand the *modification of the curvature as well as of the torsion caused by Santilli's geometrization of interior physical media.* As we shall see, this modification is the desired feature to avoid excessive approximations, such as the admission of the perpetual motion within a physical environment, which is inherent in Einstein's gravitation from its local Lorentz's symmetry.

The extension of the results to a (0.2)–rank tensor is tedious but trivial, yielding the expression

$$X^{jl}{}_{\uparrow h \uparrow k} - X^{jl}{}_{\uparrow k \uparrow h} = \hat{R}_{r}{}^{j}{}_{hk} T^{r}{}_{s} X^{sl} + \hat{R}_{r}{}^{l}{}_{hk} T^{r}{}_{s} X^{js} -$$

$$- \hat{\tau}_{h}{}^{r}{}_{k} T_{r}{}^{s} X^{jl}{}_{\uparrow s}, \tag{10.42}$$

Similarly, for contravariant isovectors and isotensors one obtains

$$X_{j \uparrow h \uparrow k} - X_{j \uparrow k \uparrow h} = -\hat{R}_{j}{}^{r}{}_{hk} T_{r}{}^{S} X_{S} - \hat{\tau}_{h}{}^{r}{}_{k} T_{r}{}^{S} X_{j \uparrow s} \tag{10.43a}$$

$$X_{jl \uparrow h \uparrow k} - X_{jl \uparrow k \uparrow h} = -\hat{R}_{j}{}^{r}{}_{hk} T_{r}{}^{s} X_{sl} - \hat{R}_{l}{}^{r}{}_{hk} T_{r}{}^{s} X_{js}$$

$$- \hat{\tau}_{h}{}^{r}{}_{k} T^{s}{}_{k} X_{jl \uparrow s}. \tag{10.43b}$$

Relations (10.42) and (10.43) will be referred to as the *Ricci–Santilli identities.*

Santilli (loc. cit.) then passes to the study of the properties of the isocurvature tensor. The following first property is an easy derivation of definition (10.39).

<u>*PROPERTY 1:*</u>

$$\hat{R}_l{}^j{}_{hk} = -\hat{R}_l{}^j{}_{kh}. \tag{10.44}$$

The second property requires some algebra, which can be derived via a simple isotopy of the conventional derivation (Lovelock and Rund (*loc. cit.*) pp. 91–92).

<u>*PROPERTY 2:*</u>

$$\hat{R}_l{}^j{}_{hk} + \hat{R}_h{}^j{}_{lh} + \hat{R}_k{}^j{}_{lh} = \hat{\tau}_l{}^j{}_{h\uparrow k} + \hat{\tau}_h{}^j{}_{kl\uparrow l} + \hat{\tau}_k{}^j{}_{l\uparrow h} +$$

$$+ \hat{\tau}_l{}^j{}_r T^r{}_s \hat{\tau}_h{}^s{}_k + \hat{\tau}_h{}^j{}_r T^r{}_s \hat{\tau}_k{}^s{}_l + \hat{\tau}_k{}^j{}_r T^r{}_s \hat{\tau}_l{}^s{}_h +$$

$$+ \hat{\tau}^j{}_{rh} \frac{\partial T^r{}_s}{\partial x^k} \hat{1}^s{}_l + \hat{\tau}^j{}_{rk} \frac{\partial T^r{}_s}{\partial x^l} \hat{1}^s{}_h + \hat{\tau}^j{}_{rl} \frac{\partial T^r{}_s}{\partial x^h} \hat{1}^s{}_k. \tag{10.45}$$

where, again, the reader should note the isotopies of the conventional terms, plus two new terms which are important physical applications indicated earlier in which the interior characteristic functions are averaged into constants.

Note that, for a symmetric isoconnection, the isotorsion is null and the above property reduces to the familiar form

$$\hat{R}_l{}^j{}_{hk} + \hat{R}_h{}^j{}_{kl} + K_k{}^j{}_{lh} = 0. \tag{10.46}$$

The third property identified by Santilli also requires some tedious but simple algebra given by an isotopy of the conventional derivation (Lovelock and Rund (*loc. cit.*), pp.92–93), which results in

<u>*PROPERTY 3:*</u>

$$(\hat{R}_j{}^l{}_{h\uparrow p} + \hat{R}_j{}^l{}_{kp\uparrow h} + \hat{R}_j{}^l{}_{ph\uparrow k}) Y_l =$$

$$= (\hat{S}_h{}^r{}_k T_r{}^s K_j{}^l{}_{sp} + \hat{S}_k{}^r{}_p T_r{}^s \hat{R}_j{}^l{}_{sh} + \hat{S}_k{}^r{}_h T_r{}^s \hat{R}_j{}^l{}_{sk}) Y_l +$$

$$+ (\hat{R}_j{}^r{}_{hk} T_r{}^l{}_{\uparrow p} + \hat{R}_j{}^r{}_{kp} T_r{}^l{}_{\uparrow h} + \hat{R}_j{}^r{}_{ph} T_r{}^l{}_{\uparrow k}) Y_l +$$

$$+ (\hat{S}_{h\ k}^{\ r} T^{l}_{r\uparrow p} + \hat{S}_{k\ p}^{\ r} T^{l}_{r\uparrow h} + \hat{S}_{p\ h}^{\ r} T^{l}_{r\uparrow k}) T\Upsilon_{j\uparrow l}, \qquad (10.47)$$

called by Santilli *isobianchi identity*, but which is now known as the *Bianchi-Santilli identity*, and which can be written in a number of equivalent forms here left to the interested reader (see an alternative expression in the next section).

Again, as it was the case for property (10.45), the Bianchi-Santilli identities (10.47) for the case of a symmetric isoconnection reduces to

$$\hat{R}_{j\ hk\uparrow p}^{\ l} + \hat{R}_{j\ kp\uparrow k}^{\ l} + \hat{R}_{j\ ph\uparrow k}^{\ l} \equiv 0. \qquad (10.48)$$

This completes the identification of all primary properties of an isocurvature tensor prior to the introduction of the isometric. Other properties, such as the *Freud identity* (Freud (1939), Pauli (1981), Yilmaz (1990)), will be studied in the next section because they require the isometric for their proper definition.

I.11: ISORIEMANNIAN GEOMETRY

In this section I shall review *Santilli's Riemannian-isotopic geometry*, or *isoriemannian geometry* for short, which is the most general possible, *nonlinear and nonlocal* geometry with a *symmetric* connection. The new geometry was introduced, apparently for the first time, in Santilli (1988d), developed in more details in Santilli (1991b) and applied to the generalization of Einstein's gravitation in Santilli (1988d), (1991d).

As predictable from the presentation of Fig. 1, Santilli conceived the study only as preparatory for the construction of the more general *Riemannian-admissible geometry*, also called *genoriemannian geometry*, namely, the yet more general, nonlinear and nonlocal geometry which can be constructed with a connection on a bimodular, affine-admissible spaces $<M>(x, <\Re>)$. The latter geometry shall be ignored in this section for brevity, and only briefly indicated in App. C.

To begin, let us perform the transition from the n-dimensional isoaffine spaces $\hat{M}(x,\hat{\Re})$ of the preceding section, to the corresponding Santilli's isospaces $\hat{M}(x,\hat{g},\hat{\Re})$ equipped with a (sufficiently smooth, real valued and nowhere singular) *symmetric isotensor* $\hat{g}_{ij}$ *of rank (0, 2)* on $\hat{M}(x,\hat{\Re})$, called by Santilli *isometric*, with a dependence on the local coordinates x, their derivatives with respect to an independent (invariant) parameter s of arbitrary order, as well as any additional

quantity needed for specific physical applications, such as the density μ of the interior physical medium considered, its temperature τ, its possible index of refraction n, etc.

$$\hat{g}_{ij}(x, \dot{x}, \ddot{x}, \mu, \tau, n, ...) = \hat{g}_{ji}(x, \dot{x}, \ddot{x}, \mu, \tau, n, ...). \tag{11.1}$$

It is easy to see that isospaces $\hat{M}(x,\hat{g},\hat{\mathfrak{R}})$ are a direct extension to an arbitrary dimension n of *Santilli's isoeuclidean spaces* $\hat{E}(r,\hat{\delta},\hat{\mathfrak{R}})$ in three-dimensions used for the construction of the *isogalilean symmetries* $\hat{G}_{\hat{\delta}}(3.1)$, as well as of *Santilli's isominkowski spaces* $\hat{M}^{III}(x,\hat{g},\hat{\mathfrak{R}})$ in (3.1)-dimension used for the construction of the *isopoincare' symmetries* $\hat{P}_{\hat{g}}(3.1)$ (Santilli (1991d)). In this section we shall continue our study of the general n-dimensional case, by keeping in mind that, from a physical viewpoint, we are primarily interested in the isoeuclidean and isominkowski subcases.

To begin, let us restrict our attention to the following isospaces.

DEFINITION 11.1 (Santilli (loc. cit.)): The "isotopic liftings" $\hat{R}(x,\hat{g},\hat{\mathfrak{R}})$ of a conventional Riemannian space $R(x,g,\mathfrak{R})$ in n-dimension (see, e.g., Lovelock-Rund (1975)) called "Riemannian-Santilli spaces" or "Santilli's isoriemannian spaces", are the n-dimensional isoaffine spaces $\hat{M}(x,\hat{\mathfrak{R}})$ equipped with a (sufficiently smooth, nowhere singular, real valued and symmetric) isometric $\hat{g} = Tg$ characterizing, first, the isofield $\hat{\mathfrak{R}}$ via the rules

$$\hat{g} = \hat{g}(x, \dot{x}, ...) = T(x, \dot{x}, ...)\, g(x)\,,\ \ \hat{g} \in \hat{R},\ \ g \in R \tag{11.2a}$$

$$\hat{\mathfrak{R}} = \mathfrak{R}\hat{1}, \qquad \hat{1} = T^{-1}, \tag{11.2b}$$

and then a symmetric isoaffine connection, called "Christoffel-Santilli symbols of the first kind"

$$\hat{\Gamma}^{1}{}_{hlk} = \tfrac{1}{2}\Big(\frac{\partial\hat{g}_{kl}}{\partial x^{h}} + \frac{\partial\hat{g}_{lh}}{\partial x^{k}} - \frac{\partial\hat{g}_{hk}}{\partial x^{l}} \Big) = \Gamma^{1}{}_{klh} \tag{11.3}$$

as well as the "Christoffel-Santilli symbols of the second kind"

$$\hat{\Gamma}^{2}{}_{h}{}^{i}{}_{k} = \hat{g}^{ij}\hat{\Gamma}^{1}{}_{hjk} = \hat{\Gamma}^{2}{}_{k}{}^{i}{}_{h} \tag{11.4}$$

where the capability for an isometric of raising and lowering the indices is understood (as in any affine space), and

$$\hat{g}^{ij} = |(\hat{g}_{rs})^{-1}|^{ij}. \tag{11.5}$$

The "Riemannian-Santilli geometry", or "Santilli's isoriemannian geometry", is the geometry of isospaces $\hat{R}(x,\hat{g},\hat{\mathfrak{R}})$.

In essence, the above definition is centered on the requirement that the alteration (also called *"mutation"* in Santilli (1978b)) $g(x) \Rightarrow T(x,\dot{x},\ddot{x},...)\, g(x) = \hat{g}(x,\dot{x},\ddot{x},...)$ of the original Riemannian metric g is characterized by an isotopic element T which is the inverse of the isounit $\hat{1}$ of the theory.

This implies that the transformation theory of the conventional Riemannian space must be lifted into the isotopic form of Sect. 10. In turn, this ensures that the isoriemannian geometry is isolinear and isolocal on $\hat{R}(x,\hat{g},\hat{\mathfrak{R}})$, although generally nonlinear and nonlocal when formulated on $R(x,g,\mathfrak{R})$, as desired.

In order to avoid insidious topological problems, the reader should be aware that both metrics $g(x)$ and $\hat{g}(x,\dot{x},\ddot{x},...)$ can be nonlinear, but the nonlocal-integral terms must all be embedded in the isotopic part T of the isometric $\hat{g}$, and cannot be admitted in the original Riemannian metric $g(x)$. This implies the embedding of all nonlocal terms in the isounit $\hat{1} = \hat{1}(x,\dot{x},\ddot{x},....)$, thus ensuring the topological consistency of the new geometry.

On physical grounds, the isotopies $R(x,g,\mathfrak{R}) \Rightarrow \hat{R}(x,\hat{g},\hat{\mathfrak{R}})$ imply that we have performed the transition from the exterior to the interior gravitational problem. Throughout the analysis of this volume, the reader should keep in mind that the isotopic elements T (or isounit $\hat{1}$) assume their conventional unit value 1 = diag. (1,1,.....,1) everywhere in the exterior of the minimal surface S° encompassing all matter of the interior problem (Sect. 1 and Fig. 1, Sect. 10), in which case $R(x,g,\mathfrak{R}) \equiv \hat{R}(x,\hat{g},\hat{\mathfrak{R}})$.

In this section we shall study the isoriemannian geometry *per se*, and without any boundary condition on the isotopic element. The condition to recover the conventional Riemannian geometry in the exterior problem will be imposed in Sect. 12.

Note that *each given gravitational theory can be subjected to an infinite number of isotopic liftings* which are expected to represent the infinite number of possible, different, interior physical media for each given total gravitational mass. This is the reason for the use the plural "isotopies".

As indicated in Definition 11.1, the introduction of a metric on an affine space implies the capability of raising and lowering the indices. The same property evidently persists under isotopy.

Given a contravariant isovector X^i on $\hat{R}(x,\hat{g},\hat{\mathfrak{R}})$, one can define its covariant form via the familiar rule

$$X_i = \hat{g}_{ij} X^j. \tag{11.6}$$

Similar conventional rules apply for the lowering of the indices of all other quantities.

It is easy to see that the inverse of g_{ij}, Eq.s (11.5), is a bona-fide contravariant isotensor of rank (2. 0). Given a covariant isovector X_i on $\hat{R}(x,\hat{g},\hat{\mathfrak{R}})$, its contravariant form is then defined by

$$X^i = \hat{g}^{ij} X_j. \tag{11.7}$$

Rules (11.6) and (11.7) can then be used to raise or lower the indices of an arbitrary isotensor of rank (r. s).

From the definition of the Christoffel–Santilli symbols of the first kind, Eq.s (11.3), we have

$$\frac{\partial \hat{g}_{hl}}{\partial x^k} = \hat{\Gamma}^l{}_{hlk} + \hat{\Gamma}^l{}_{lhk}, \tag{11.8}$$

and

$$\hat{g}_{hl\uparrow k} = \frac{\partial g_{hl}}{\partial x^k} - \hat{\Gamma}^l{}_{hlk} - \hat{\Gamma}^l{}_{lhk}. \tag{11.9}$$

Thus,

$$\hat{g}_{hl\uparrow k} \equiv 0, \qquad \hat{g}^{hl}{}_{\uparrow k} \equiv 0, \tag{11.10}$$

We reach in this way the following

LEMMA 11.1 (Ricci–Santilli Lemma; Santilli (1988d), (1991b, d)) : All (sufficiently smooth, and regular) isotopic liftings of the Riemannian geometry preserve the vanishing character of the covariant derivative of the isometrics.

In different terms, the familiar property of the Riemannian geometry

$$g_{ij\,|\,k} = 0 \tag{11.11}$$

is a true geometric axiom because it is invariant under all infinitely possible isotopies. As shown by Santilli (see below), this property is not shared by all

quantities of current use in gravitation.

The isotransformation law of the isometric $\hat{g}$ is given by

$$\hat{g}_{ij}(x, \dot{x},..) = \frac{\partial \bar{x}^p}{\partial x^i} \bar{T}_p{}^r(\bar{x}, \dot{\bar{x}},..) \hat{\bar{g}}_{rs}(\bar{x}, \dot{\bar{x}},..)) \bar{T}^s{}_q(\bar{x}, \dot{\bar{x}},...) \frac{\partial \bar{x}^q}{\partial x^j}. \tag{11.12}$$

where the isotopic elements $T^r{}_p$ in the r.h.s. are again computed in the new coordinate system as in Eq.s (10.32).

By repeating the conventional procedure (see Lovelock and Rund (*loc. cit.*)), pp. 78–70) under isotopy, Santilli obtains the following expression for the *Christoffel–Santilli symbol of the first kind*

$$\hat{\Gamma}^l{}_{hlk} = \tfrac{1}{2}\Big(\frac{\partial \hat{g}_{kl}}{\partial x^h} + \frac{\partial \hat{g}_{lh}}{\partial x^k} - \frac{\partial \hat{g}_{hk}}{\partial x^l} \Big) = \tag{11.13}$$

$$= \hat{g}_{jp}\, T^j{}_r \frac{\partial^2 x^r}{\partial x^h \partial x^k} T^p{}_s \frac{\partial x^s}{\partial x^l} + \frac{\partial \hat{\bar{g}}_{jp}}{\partial \bar{x}^m} T^j{}_r\, T^p{}_s \Big(\frac{\partial \bar{x}^r}{\partial x^h} \frac{\partial \bar{x}^s}{\partial x^k} \frac{\partial \bar{x}^m}{\partial x^l} +$$

$$+ \frac{\partial \bar{x}^r}{\partial x^k} \frac{\partial \bar{x}^s}{\partial x^l} \frac{\partial \bar{x}^m}{\partial x^h} - \frac{\partial \bar{x}^r}{\partial x^l} \frac{\partial \bar{x}^s}{\partial x^h} \frac{\partial \bar{x}^m}{\partial x^k} \Big) +$$

$$+ \tfrac{1}{2}\hat{g}_{jp}\, T^p{}_s \Big[\frac{\partial T^j{}_r}{\partial x^l} \Big(\frac{\partial \bar{x}^r}{\partial x^h} \frac{\partial \bar{x}^s}{\partial x^k} + \frac{\partial \bar{x}^s}{\partial x^h} \frac{\partial \bar{x}^r}{\partial x^k} \Big) +$$

$$+ \frac{\partial T^j{}_r}{\partial x^h} \Big(\frac{\partial \bar{x}^r}{\partial x^k} \frac{\partial \bar{x}^s}{\partial x^l} + \frac{\partial \bar{x}^s}{\partial x^k} \frac{\partial x^r}{\partial x^l} \Big) - \frac{\partial T^j{}_r}{\partial x^k} \Big(\frac{\partial \bar{x}^r}{\partial x^l} \frac{\partial \bar{x}^s}{\partial x^h} + \frac{\partial \bar{x}^s}{\partial x^l} \frac{\partial \bar{x}^r}{\partial x^h} \Big)$$

with a number of alternative formulations and simplifications, e.g., for diagonal isotopic elements T, which are left to the interested reader for brevity.

The contravariant isometric $\hat{g}^{ij}$ evidently verifies the isotransformation laws

$$\hat{g}^{ij}(x, \dot{x},...) = \frac{\partial x^i}{\partial \bar{x}^r} T^r{}_p(\bar{x}, \dot{\bar{x}},...) \hat{\bar{g}}^{pq}(\bar{x}, \dot{\bar{x}},..) T^s{}_q(\bar{x}, \dot{\bar{x}},..) \frac{\partial x^j}{\partial \bar{x}^s}. \tag{11.14}$$

In order to proceed with our review, we need the following

DEFINITION 11.2 (Santilli (loc. cit.)): Given an isoriemannian space $\hat{R}(x,\hat{g},\hat{\mathfrak{R}})$ in n dimension, the "isocurvature tensor" is given by

$$\hat{R}_{l\,hk}^{\;j} = \frac{\partial \Gamma^{2\,j}_{l\,h}}{\partial x^k} - \frac{\partial \Gamma^{2\,j}_{l\,k}}{\partial x^h} + \\ + \Gamma^{2\,j}_{m\,k} T^m{}_r \Gamma^{2\,r}_{l\,h} - \Gamma^{2\,j}_{m\,h} T^m{}_r \Gamma^{2\,r}_{l\,k} + \\ + \Gamma^{2\,j}_{r\,h} \frac{\partial T^r{}_s}{\partial x^k} \hat{1}^s{}_l - \Gamma^{2\,j}_{r\,k} \frac{\partial T^r{}_s}{\partial x^h} \hat{1}^s{}_l \tag{11.15}$$

and can be rewritten

$$\hat{R}_{l\,hk}^{\;j} = \tfrac{1}{2}\hat{g}^{jp}\left(\frac{\partial \hat{g}_{ph}}{\partial x^k \partial x^l} - \frac{\partial \hat{g}_{pk}}{\partial x^h \partial x^l} + \frac{\partial \hat{g}_{lk}}{\partial x^h \partial x^j} - \frac{\partial \hat{g}_{lh}}{\partial x^k \partial x^j}\right) + \\ + \hat{g}^{jp}\left(\hat{\Gamma}^{l}_{prh} T^r{}_s \hat{\Gamma}^{2\,s}_{l\,h} - \hat{\Gamma}^{l}_{prk} T^r{}_s \Gamma^{2\,s}_{l\,k}\right) + \\ + \Gamma^{2\,j}_{r\,h} \frac{\partial T^r{}_s}{\partial x^k} \hat{1}^s{}_l - \Gamma^{2\,j}_{r\,k} \frac{\partial T^r{}_s}{\partial x^h} \hat{1}^s{}_l, \tag{11.16}$$

the "Ricci-Santilli tensor" is given by

$$\hat{R}_{lh} = \hat{R}_{l\,hj}^{\;j} = g^{ij} \hat{R}_{lihj}; \tag{11.17}$$

the "Einstein-Santilli tensor" is given by

$$\hat{G}_i^{\;j} = \hat{R}_i^{\;j} - \tfrac{1}{2}\delta_i^{\;j}\hat{R}; \tag{11.18}$$

and the "completed Einstein-Santilli tensor" is given by

$$\hat{S}_i^{\;j} = \hat{R}_i^{\;j} - \tfrac{1}{2}\delta_i^{\;j}\hat{R} - \tfrac{1}{2}\delta_i^{\;j}\hat{\Theta}; \tag{11.19}$$

where $\hat{R}$ is the "isocurvature isoscalar"

$$\hat{R} = \hat{R}^{i}_{\ i} = \hat{g}^{ij}\hat{R}_{ij}, \tag{11.20}$$

and $\hat{\theta}$ is the "isotopic isoscalar"

$$\hat{\theta} = \hat{g}^{jh}\hat{g}^{lk}(\hat{\Gamma}^{1}_{\ rjk}T^{r}_{\ s}\hat{\Gamma}^{2\ s}_{\ l\ h} - \hat{\Gamma}^{1}_{\ rjh}T^{r}_{\ s}\hat{\Gamma}^{2\ s}_{\ l\ k}) =$$

$$= \hat{\Gamma}^{1}_{\ rjk}T^{r}_{\ s}\hat{\Gamma}^{2\ s}_{\ l\ h}(\hat{g}^{jh}\hat{g}^{lk} - \hat{g}^{jk}\hat{g}^{lh}). \tag{11.21}$$

We are now equipped to review Santilli (*loc. cit.*) isotopies of the various properties of the Riemannian geometry as available in various textbooks on gravitation. From definition (11.16) we readily obtain

PROPERTY 1: Antisymmetry of the last two indeces of the isocurvature tensor:

$$\hat{R}_{l\ hk}^{\ j} = - \hat{R}_{l\ kh}^{\ j}. \tag{11.22}$$

The specialization of properties (10.45) to the case at hand easily implies the following

PROPERTY 2: Vanishing of the totally antisymmetric part of the isocurvature tensor

$$\hat{R}_{l\ hk}^{\ j} + \hat{R}_{h\ kl}^{\ j} + \hat{R}_{k\ lh}^{\ j} = 0, \tag{11.23}$$

or, equivalently,

$$\hat{R}_{lmhk} + \hat{R}_{hmkl} + \hat{R}_{kmlh} = 0. \tag{11.24}$$

The use of property (11.22) and Lemma 10.1 then yields

PROPERTY 3: Antisymmetry in the first two indices of the isocurvature tensor

$$\hat{R}_{jlhk} = \hat{R}_{ljhk}, \tag{11.25}$$

or, equivalently,

$$\hat{R}_{ljhk} = \hat{R}_{hklj}. \qquad (11.26)$$

From Definition (11.15) and the use of the Ricci-Santilli Lemma, after tedious but simple calculations, Santilli obtains the following

PROPERTY 3: Bianchi-Santilli identity

$$\hat{R}_{l\,hk\uparrow p}^{\ j} + \hat{R}_{l\,ph\uparrow k}^{\ j} + \hat{R}_{l\,kp\uparrow h}^{\ j} = \hat{S}_{l\,hkp}^{\ j}, \qquad (11.27)$$

where

$$\begin{aligned}\hat{S}_{l\,hkp}^{\ j} = {} & \hat{\Gamma}^{2\ j}_{\ rh}(T^r_{\ s|k}\hat{\Gamma}^{2\ s}_{\ lp} - T^r_{\ s\uparrow p}\hat{\Gamma}^{2\ s}_{\ lk}) + \\ & + \hat{\Gamma}^{1\ j}_{\ rp}(T^r_{\ s|h}\hat{\Gamma}^{2\ s}_{\ lk} - T^r_{\ s|k}\hat{\Gamma}^{2\ s}_{\ lh}) + \\ & + \hat{\Gamma}^{1\ j}_{\ rk}(T^r_{\ s\uparrow p}\hat{\Gamma}^{2\ s}_{\ lh} - T^r_{\ s|h}\hat{\Gamma}^{\ s}_{\ lp}) + \\ & + \hat{\Gamma}^{2\ j}_{\ rh}(\hat{Q}_{k\ l\uparrow p}^{\ r} - \hat{Q}_{p\ l\uparrow k}^{\ r}) + \hat{\Gamma}^{2\ j}_{\ rp}(\hat{Q}_{h\ l\uparrow k}^{\ r} - \hat{Q}_{k\ l\uparrow h}^{\ r}) + \\ & + \hat{\Gamma}^{2\ j}_{\ rk}(\hat{Q}_{p\ l\uparrow h}^{\ r} - \hat{Q}_{h\ r\uparrow p}^{\ r}),\end{aligned} \qquad (11.28)$$

and

$$\hat{Q}_{k\ l\uparrow p}^{\ r} = \left(\frac{\partial T^r_{\ s}}{\partial x^k}\hat{1}^s_{\ l}\right)\uparrow p. \qquad (11.29)$$

For isotopic liftings independent from the local coordinates (but dependent on the velocities and other variables, as it is generally the case, or for the characteristic functions of the interior physical medium averaged into constants, isodifferential property (11.27) assumes the simpler form

$$\hat{R}_{l\,hk\uparrow p}^{\ j} + \hat{R}_{l\,ph\uparrow k}^{\ j} + \hat{R}_{l\,kp|h}^{\ j} \equiv 0. \qquad (11.30)$$

The Bianchi-Santilli identity can also be equivalently written in the general case

$$\hat{R}_{ijhk\uparrow p} + \hat{R}_{ijph\uparrow k} + \hat{R}_{ijkp\,|\,h} = \hat{S}_{ijhkp}, \tag{11.31}$$

where the $\hat{S}$-term is that defined by Eq.s (11.28), with the reduced form for the isotopies not dependent on the local coordinates or constant

$$\hat{R}_{ijhk\uparrow p} + \hat{R}_{ijph\uparrow k} + \hat{R}_{ijkp\uparrow h} \equiv 0. \tag{11.32}$$

We now consider the isotopic liftings of *Freud identity* which was originally identified by Freud (1949), first reviewed by Pauli (1958), and then forgatten for a long time. The identity was "rediscovered" by Yilmaz (1990), Carmeli *et al.* (1990), and finally studied in detail by Rund (1991). Santilli (1991b,d) followed Rund's treatment, and reached the following property we shall call

PROPERTY 5: Freud-Santilli identity

$$\hat{U}'^{k}{}_{j} + \hat{G}'^{k}{}_{j} = \frac{\partial \hat{V}'^{kl}{}_{i}}{\partial x^{l}}, \tag{11.33}$$

where

$$\hat{V}'^{kl}{}_{j} = \tfrac{1}{2}\hat{\Delta}^{\frac{1}{2}}\{\hat{g}^{rs}(\delta^{k}{}_{j}\Gamma^{2}{}_{r}{}^{l}{}_{s} - \delta^{l}{}_{j}\Gamma^{2}{}_{r}{}^{k}{}_{s}) +$$

$$+(\delta^{l}{}_{j}\hat{g}^{kr} - \delta^{k}{}_{j}\hat{g}^{lr})\hat{\Gamma}^{2}{}_{r}{}^{s}{}_{s} + \hat{g}^{lr}\Gamma^{2}{}_{j}{}^{k}{}_{r} - \hat{g}^{kr}\Gamma^{2}{}_{j}{}^{l}{}_{r}), \tag{11.34a}$$

$$\hat{U}'^{k}{}_{j} = \tfrac{1}{2}\hat{\Delta}^{\frac{1}{2}}\left(\frac{\partial \hat{G}}{\partial \hat{g}^{lm}{}_{\uparrow k}}\hat{g}^{lm}{}_{\uparrow j} - \delta^{k}{}_{j}\hat{G}\right), \tag{11.34b}$$

$$\hat{G} = \hat{g}^{jk}(\Gamma^{2}{}_{j}{}^{p}{}_{s}\,T^{q}{}_{p}\Gamma^{2}{}_{k}{}^{s}{}_{q} - \Gamma^{2}{}_{j}{}^{p}{}_{k}\,T^{q}{}_{p}\,\Gamma^{2}{}_{q}{}^{s}{}_{s}), \tag{11.34c}$$

$$\hat{G}'^{k}{}_{j} = \hat{\Delta}^{\frac{1}{2}}\hat{G}^{k}{}_{j}, \quad \hat{\Delta}^{\frac{1}{2}} = \sqrt{\hat{g}}. \tag{11.34d}$$

A major result of Rund's (loc. cit.) analytis is that *the conventional Freud identity holds for all symmetric and nonsingular metric on a conventional Riemannian space of dimension higher than one.* The same property evidently persist unde Sanbtilli's isotopies. Thus, *Property 5 is automatically satisfied for all symmetric and nonsingular metrics on isoriemannian spaces of dimension higher than one.*

We are now in a position to identify the most salient consequences of

Santilli's isoriemannian geometry. First, it is an instructive exercise for the reader interested in acquiring a technical knowledge of the isotopies of the Riemannian geometry to prove the following important property.

LEMMA II.2 (Santilli (loc. cit.)): Einstein's tensor

$$G^i_j = R^i_j - \tfrac{1}{2}\delta^i_j R \tag{II.35}$$

does not preserve under isotopies the vanishing value of its covariant divergence (contracted Bianchi identity)

$$G^i_{j|i} = R^i_{j|i} - \tfrac{1}{2}\delta^i_j R_{|i} \equiv 0, \tag{II.36}$$

that is, the isoinsteinian tensor (II.18) violates property (II.36),

$$\hat{G}^i_{j\uparrow i} = \hat{R}^i_{j\uparrow i} - \tfrac{1}{2}\delta^i_j \hat{R}_{\uparrow i} \not\equiv 0, \tag{II.37}$$

Therefore, Einstein's tensor does not possess an axiomatically complete structure, and the contracted Bianchi identity does not constitute an axiom of the Riemannian geometry.

This rather unexpected occurrence has rather deep meaning for the now vexing, open problem of the possible source to the field equations in vacuum (see next section).

The following important additional property can also be proved via tedious but simple calculations from isodifferential property (II.27).

LEMMA II.3 (loc. cit.): The completed Einstein-Santilli tensor (II.19) does possess an identically null isocovariant isodivergence, i.e.,

$$\hat{S}^i_{j\uparrow j} = (\hat{R}^i_j - \tfrac{1}{2}\delta^i_j \hat{R} - \tfrac{1}{2}\delta^i_j \hat{\Theta})_{\uparrow i} \equiv 0. \tag{II.38}$$

hereon referred to as the "completed and contracted Bianchi-Santilli identity".

By reinspceting the above findings, we can say that Einstein's tensor G^i_j is not "axiomatically complete" because it does not possess properties that are invariant under all infinitely possible isotopies. However, if Einstein's tensor is "completed" by adding a suitable tensor with null covariant divergence, then it is turned into a true axiomatic form invariant under isotopy.

It is interesting to note that *the Freud identity is a true geometric axiom of the Riemannian geometry in the sense that it persists under isotopies, while the contracted Bianchi identity is not an axiom of the Riemannian geometry, evidently because not preserved by isotopies.*

By following again Santilli (1991b), let us first identify the implications of the above findings for the *conventional* theory of gravitation, and then study their isotopies. For this purpose, we introduce the following

DEFINITION 11.3 (loc. cit.): The "completed Einstein's tensor" on $R(x,g,\Re)$ is given by the expression

$$S^i_{\ j} = R^i_{\ j} - \tfrac{1}{2}\delta^i_{\ j} R - \tfrac{1}{2}\delta^i_{\ j}\,\theta, \tag{11.39}$$

where $R^i_{\ j}$ is the conventional Ricci tensor, R is the conventional curvature scalar and θ is given by the isotopic quantity $\hat{\theta}$, Eq. (11..21), for $T = I$, i.e.,

$$\theta = g^{jh}\, g^{lk}\,(\Gamma^{1}{}_{rjk}\,\Gamma^{2\,r}{}_{l\,h} - \Gamma^{1}{}_{rjh}\,\Gamma^{2\,r}{}_{l\,k}) =$$

$$= \Gamma^{1}{}_{rjk}\,\Gamma^{2\,r}{}_{l\,h}\,(g^{jh}\, g^{lk} - g^{jk}\, g^{lh}). \tag{11.40}$$

But, the covariant derivatives of the θ-quantity are identically null (from the conventional Ricci Lemma). We therefore have the following

COROLLARY 11..2.1 (loc. cit.): Einstein's tensor can be axiomatically completed by substracting the term $\tfrac{1}{2}\delta^{i}_{\ j}\,\theta$ with null covariant derivatives as per Definiton 11.3, while preserving the null value of the covariant divergence, i.e.,

$$(S^i_{\ j\uparrow i})_{T=I} = (R^i_{\ j} - \tfrac{1}{2}\delta^i_{\ j} R - \tfrac{1}{2}\delta^i_{\ j}\,\theta)_{|i} \equiv (R^i_{\ j} - \tfrac{1}{2}\delta^i_{\ j} R)_{\uparrow i} \equiv 0. \tag{11.41}$$

which is called the "completed and contracted Bianchi identity".

The axiomatic structure which can be subjected to a consistent lifting is therefore the generalized tensor (11.39), and not Einstein's tensor.

It should be recalled that Santilli's *"completed Einstein's tensor"* has no relationship to the *"modified Einstein's tensor"* with the cosmological constant Λ, i.e., the familiar form (see, e.g., Pauli (1958))

$$\hat{G}^i_j = R^i_j - \tfrac{1}{2}\delta^i_j R + \delta^i_j \Lambda. \tag{11.42}$$

for numerous reasons. First, Λ is a constant in quantity (11.42), while θ is a scalar function in Eq.s (11.39). Secondly, tensor (11.42) leads to a static universe, as well known, while this is not the case for Santilli's completion of Einstein's tensor, as we shall see. Third, the modified tensor (11.42) also does not possess sufficient generality to constitute a geometric axiom invariant under isotopies.

At this point, it is important to identify the implications for the gravitational equations prior to the addition of gravitational sources (to be done shortly in this section).

A repetition of the analysis by Lovelock and Rund ((*loc. cit.*), p. 313 and the Theorem of p. 321) for the completed Einstein's tensor leads to the following

THEOREM 11.1 (Santilli (loc. cit.)): In a (conventional) four-dimensional Riemannian space $R(x,g,\Re)$ the most general possible, axiomatically complete Euler-Lagrange equations

$$\hat{E}^{ij} = 0, \tag{11.43}$$

verifying the properties

$$\hat{E}^{ij} = \hat{E}^{ji}, \qquad \hat{E}^{ij}{}_{|j} = 0, \tag{11.44a}$$

$$\hat{E}^{ij} = \hat{E}^{ij}(g_{ij}, g_{ij,k}, g_{ij,kl}), \quad g_{ij,k} = \partial g_{ij} / \partial x^k, \text{ etc.} \tag{11.44b}$$

(where the latter property also expresses the lack of source), are characterized by the variational principle

$$\delta\hat{A} = \delta\int L(g_{ij}, g_{ij,k}, g_{ij,kl})\, dx =$$

$$= \delta\int \Delta^{\frac{1}{2}}[\lambda(R+\theta) - 2\Lambda] = 0, \qquad \Delta^{\frac{1}{2}} = \sqrt{g} \tag{11.45}$$

and read

$$\hat{E}^{ij} = \Delta^{\frac{1}{2}}\{\lambda[R^{ij} - \tfrac{1}{2}g^{ij}(R+\theta)] + \Lambda g^{ij}\} = 0, \tag{11.46}$$

where R is the curvature scalar and θ is quantity (11.40).

The reader will recognize in the above theorem the cosmological constant Λ, as is well as its differentiations from our θ-quantity. The reader will also see the

difference of the gravitational equations (11.50) with the corresponding Einsteinian form.

The isotopies of the above property can be readily done, via the methods illustrated earlier, thus reaching the following

THEOREM 11.2: In a four-dimensional isoriemannian space $\hat{R}(x,\hat{g},\hat{\mathfrak{R}})$, the most general possible Euler-Lagrange equations

$$\hat{E}^{ij} = 0, \tag{11.47}$$

verifying the properties

$$\hat{E}^{ij} = \hat{E}^{ji}, \quad \hat{E}^{ij}{}_{\hat{|}j} = 0, \tag{11.48a}$$

$$\hat{E}^{ij} = \hat{E}^{ij}(\hat{g}_{ij}, \hat{g}_{ij,k}, \hat{g}_{ij\,kl}), \quad \hat{g}_{ij,k} = \partial\hat{g}_{ij} / \partial x^k, \quad \text{etc.} \tag{11.48b}$$

where the latter property denotes absence of sources, are characterized by the variational principle

$$\delta\hat{A} = \delta\int \hat{L}(\hat{g}_{ij}, \hat{g}_{ij,k}, \hat{g}_{ij,kl})\, dx =$$
$$= \delta\int \hat{\Delta}^{\frac{1}{2}}[\lambda(\hat{R} + \hat{\theta}) - 2\Lambda] = 0, \quad \hat{\Delta}^{\frac{1}{2}} = (\hat{g})^{\frac{1}{2}} \tag{11.49}$$

and read

$$\hat{E}^{ij} = \hat{\Delta}^{\frac{1}{2}}\{\lambda[\hat{R}^{ij} - \tfrac{1}{2}\hat{g}^{ij}(\hat{R} + \hat{\theta})] + \Lambda\hat{g}^{ij}\}, \tag{11.50}$$

where $\hat{R}$ is the isocurvature isoscalar (11..20) and $\hat{\theta}$ is the isotopic isoscalar (11..21).

This completes our review of the conventional and isotopic Riemannian geometry without sources.

The most general possible formulation of gravitation on an isoriemannian manifold with sources, possesses the following structure.

THEOREM 11..5 (Santilli (loc. cit.)): In a four-dimensional isoriemannian space $\hat{R}(x,\hat{g},\hat{\mathfrak{R}})$, the most general possible Euler-Lagrange equations

$$\hat{E}^{ij} = 0, \tag{11.51}$$

verifying the properties: 1) symmetric condition on the Euler–Lagrange tensor

$$\hat{E}^{ij} = \hat{E}^{ji}, \tag{11.52}$$

2) the contracted Bianchi–Santilli identity

$$\hat{E}^{il}{}_{\uparrow l} = 0, \tag{11.53}$$

and 3] the Freud–Santilli identity

$$\hat{U}^{k}{}_{j} + \hat{G}^{k}{}_{j} = \frac{\partial \hat{V}^{kl}}{\partial x^{l}}, \tag{11.54}$$

are characterized by the variational principle

$$\begin{aligned} \delta\hat{A} &= \delta\int \hat{L}(\hat{g}_{ij}, \hat{g}_{ij,k}, \hat{g}_{ij,kl}, \tau_{ij}, t_{ij})\, dx = \\ &= \delta\int \hat{\Delta}^{\frac{1}{2}}[\lambda(\hat{R} + \hat{\theta}) + 2\Lambda + \rho(\hat{\tau} + \hat{t})]\, dx = \\ &= \delta\int \hat{\Delta}^{\frac{1}{2}}[\lambda\hat{R} + 2\Lambda + \rho(\hat{T} + \hat{t})]\, dx = 0, \end{aligned} \tag{11.55}$$

where λ, Λ, and ρ are constants, $\hat{t}$ is the isotopic generalization of Yilmaz (1979) stress–energy tensor, and

$$\hat{T} = \hat{\tau} + \lambda\hat{\theta}/\rho, \tag{11.56}$$

is the source tensor. For the case $\lambda = \rho = 1$ and $\Lambda = 0$, the Euler–Lagrange equations are given by

$$\hat{E}^{ij} = \hat{R}^{ij} - \tfrac{1}{2}\hat{g}^{ij}\hat{R} - \tfrac{1}{2}\hat{g}^{ij}\hat{\theta} - \hat{\tau}^{ij} - \hat{t}^{ij} = 0, \tag{11.57}$$

or, equivalently,

$$\hat{G}^{ij} = \hat{R}^{ij} - \tfrac{1}{2}\hat{g}^{ij}R = \hat{T}^{ij} + \hat{t}^{ij}, \tag{11.58}$$

Throughout the analysis of these sections we have often considered interior trajectories of "nonlagrangian" type. It is important to understand that this term is referred to the lack of analytic representations in terms of a *first-order Lagrangian*, i.e., a Lagrangian L depending at most on the first order derivatives of

the variables, $L = L(s, x, \dot{x})$. In this case the Euler–Lagrange equations are of second-order.

The theory of Lagrangians of order higher than the first (with Euler–Lagrange equations of order higher than the second), even though quite intriguing, implies a rather deep revision of the analytic mechanics, e.g., for the construction of the corresponding "Hamiltonian".

A first way to understand the nonlagrangian character of Santilli's isoriemannian geometry, is by recalling that the "Lagrangian" equivalent of the Birkhoffian mechanics is precisely of the second order (Santilli (1982a)).

The generally nonlagrangian character of the geometry under consideration is then made clear by the following

COROLLARY 11.5.1 (loc.. cit.): The Lagrangians of Theorem 11.5 are first–order in the metric tensor, $L = L(g_{ij}, \hat{g}_{ij,k}, \hat{g}_{ij,kl})$, *but generally of the second– or higher–order in the coordinate derivatives,* $L = L(s, \dot{x}, \ddot{x}, ...)$.

Euler–Lagrange equations of order higher than the second are avoided in the isoriemannian geometry because all derivative terms are embedded in the isometric of the theory, while the Euler–Lagrange equations are computed precisely with respect to such isometric, and not with respect to the local variables and their derivatives, as in the conventional case.

The analysis of this section is completed in the next section with the notions of isoparallel transport and isogeodesics.

We close this section with a few complementary aspects. As well known, a most important system of local coordinates is that introduced by Riemannian (1868) with the name of *"normal coordinates"*, say,

$$x^i \Rightarrow y^{\circ i}(x), \tag{11.60}$$

under which the Riemannian space $R(x,g,\mathfrak{R})$ is locally flat. In different terms, the normal coordinates are such that, in the neighborhood of the point $P^\circ = (y^\circ)$, all coefficients of the connection $\Gamma^2{}_\rho{}^\mu{}_\sigma$ are identically null,

$$\Gamma^2{}_r{}^i{}_s(y^\circ) \equiv 0. \tag{11.61}$$

Moreover, it has been proved in the literature that a system of normal coordinates always exists for all affine spaces with a *symmetric* connection. We can therefore introduce the following

DEFINITION 11.4 (Santilli (loc. cit.)): The "isonormal coordinates" of an

isoriemannian n-dimensional space $\hat{R}(x,\hat{g},\hat{\Re})$ are the coordinates $y^{\circ\, i}(x)$ such that, in the neighborhood of a point $y^{\circ\, i}$, all isoconnection coefficients are identically null

$$\hat{\Gamma}^{2\ i}_{\ r\, s}\,(y^{\circ}) = 0. \tag{11.62}$$

Normal coordinates have a fundamental physical meaning in conventional gravitational theories, because they allow the identification of the *local Lorentz frames.*

In the transition to an isotopic formulation of the Riemannian geometry, we encounter another difference with fundamental physical implications.

LEMMA 11.6 (loc. cit.): The metric holding in the neighborhood of a point of the isonormal coordinates of an isoriemannian space is isominkowskian with null isocurvature.

PROOF: Suppose that the transformations $x \Rightarrow y^{\circ}(x)$ are such to eliminate the space-dependence of the transformed isoconnection coefficients. Then, Eq.s (11.86) hold, but the local metric remains generally dependent on the derivatives $\dot{y}$, $\ddot{y}$, and other quantities, thus being of isominkowskian type. The lack of isocurvature follows from the lack of local dependence on the coordinates. Q.E.D.

Stated differently, in the conventional case, the connection coefficient can only depend on the local coordinates, $\Gamma^{2\ i}_{\ r\, s} = \Gamma^{2\ i}_{\ r\, s}(x)$. The recovering of the Euclidean metric δ or of the Minkowskian metric η under local coordinates then follows.

In the isotopic case, the isoconnection coefficients depend on the local coordinates x as well as all possible (or otherwise needed) derivatives and other quantities, $\hat{\Gamma}^{2\ i}_{\ r\, s} = \hat{\Gamma}^{2\ i}_{\ r\, s}(x, \dot{x}, \ddot{x}, \mu, \tau, n, ...)$. Their transformation under normal coordinates then eliminates the coordinate dependence of the metric, but generally leaves the dependence on the remaining variables, and we shall write

$$\bar{g}^{ij} = \frac{\partial y^{i}}{\partial x^{r}}\, T^{r}_{\ k}(x,\dot{x},\ddot{x},..)\, g^{kl}(x,\dot{x},\ddot{x},...)\, T^{s}_{\ l}(x\dot{x},\ddot{x},..)\, \frac{\partial y^{j}}{\partial x^{s}} =$$

$$= \bar{g}^{ij}\,(\dot{y}, \ddot{y},...). \tag{11.63}$$

Needless to say, coordinate transformations of an isoriemannian manifold

$$x^{i} \Rightarrow w^{i}(x, \dot{x}, \ddot{x},...), \tag{11.64}$$

admitting the Minkowskian metric may indeed exist, but they are generally nonlinear and nonlocal. In fact, for the case in which the Riemannian geometry generalizes the Euclidean setting with metric $\delta = \mathrm{diag.}\,(1,1,...,1)$, transformation (4.88) via rule (4.87) would imply

$$x^i\,\hat{g}_{ij}(x, \dot{x}, \ddot{x},..)\,x^j \;=\; w^r\,\delta_{rs}\,w^s, \tag{11.65}$$

with similar results for the case of the Minkowski metric (see Chapter V of Santilli (1991d)). Needless to say, the latter coordinates are considerably more difficult to identify than the isonormal coordinates, although their existence is not excluded here.

The central point remains that, in the isotopic case, reduction (11.65) *is not* necessarily implied by the geometric conditions (11.62). The local isotopic metric (11.63) then persists as the geometrically natural case.

As now familiar, we have initially considered a conventional gravitational theory on $R(x,g,\mathfrak{R})$ which, as well known, has *null torsion*, and have reached an infinite family of isotopies all of which also have a *null isotorsion* on $\hat{R}(x,\hat{g},\hat{\mathfrak{R}})$. In fact, the original symmetric connection $\Gamma2_h{}^s{}_k$ has been lifted into an infinite family of isoconnections which are also symmetric

$$T_h{}^s{}_k = \Gamma2_h{}^s{}_k - \Gamma2_k{}^s{}_h \equiv 0 \;\Rightarrow\; \hat{T}_h{}^s{}_k \;=\; \hat{\Gamma}2_h{}^s{}_k - \hat{\Gamma}2_k{}^s{}_h \equiv 0. \tag{11.66}$$

However, *the null value of torsion occurs at the level of Santilli's geometrical isospaces* $\hat{R}(x,\hat{g},\hat{\mathfrak{R}})$ *which are not the physical space of the experimenter*, the latter remaining the conventional space-time in vacuum (see Chapter IV for details of Santilli (1991d)).

The physical issue whether or not the isotopies of Einstein's gravitation have a non-null torsion must therefore be inspected in the physical space and not in the geometrical isospace.

This can be easily done, e.g., by rewriting the isocovariant derivative of an isovector on $\hat{R}(x,\hat{g},\hat{\mathfrak{R}})$ as a conventional covariant derivative in the ordinary space $R(x,g,\mathfrak{R})$, i.e.,

$$X^i{}_{\uparrow k} \;=\; \frac{\partial X^i}{\partial x^k} \;+\; \hat{\Gamma}2_h{}^i{}_k\,T^h{}_r\,X^r \;\equiv$$

$$\partial X^i$$

$$\equiv X^{i}{}_{|k} = \frac{}{\partial x^{k}} + \hat{\Gamma}^{2}{}_{r}{}^{i}{}_{k}\, X^{i}, \tag{11.67a}$$

$$\hat{\Gamma}^{2}{}_{r}{}^{i}{}_{k} = \hat{\Gamma}^{2}{}_{h}{}^{i}{}_{k}\, T^{h}{}_{r}. \tag{11.67b}$$

It is then evident that, starting with a symmetric isoconnection $\hat{\Gamma}_{h}{}^{i}{}_{k}$ on $\hat{R}(x,\hat{g},\mathfrak{R})$, the corresponding connection $\hat{\Gamma}_{r}{}^{i}{}_{k}$ on $R(x,g,\mathfrak{R})$ is no longer necessarily symmetric, and we have the following

LEMMA 11.7: The isotopic liftings $\Gamma^{2}{}_{h}{}^{i}{}_{k} \Rightarrow \hat{\Gamma}^{2}{}_{h}{}^{i}{}_{k}$ *of a symmetric connection* $\Gamma^{2}{}_{h}{}^{i}{}_{k}$ *on a Riemannian space* $R(x,g,\mathfrak{R})$ *into an infinite family of isotopic connections* $\hat{\Gamma}^{2}{}_{h}{}^{i}{}_{k}$ *on isoriemannian spaces* $\hat{R}(x,\hat{g},\hat{\mathfrak{R}})$ *of the same dimension, imply that the isospace always possesses a null isotorsion, but, when the isotopies are projected into the original space, a non-null torsion generally occurs.*

The above property was first reached by Gasperini (1984a, b, c) via the isotopy of Einstein's gravitation in the language of conventional differential forms on a conventional Riemannian space. The geometrization of the property into a symmetric isotorsion on an isoriemannian space was achieved by Santilli (1988d), (1991b).

At this point the advances in torsion made by Rapoport-Campodonico ((1991) and quoted papers) become applicable to Santilli's *interior* gravitation. We regret the inability to review these studies and reformulate them in terms of Santilli's null isotorsion.

Let us recall that any nonlinear and nonlocal theory can always be identically written in an isolinear and isolocal form (Sect. 3). By reversing the proof of Lemma 11.7, it is then easy to prove the following

COROLLARY 11.7.1: Under sufficient continuity and regularity conditions, any gravitational theory on a conventional affine space $R(x,\mathfrak{R})$ *with non-null torsion, can always be written in an identical form on a suitable isoaffine space* $\hat{R}(x,\hat{\mathfrak{R}})$ *of the same dimension with an identically null isotorsion.*

Let us recall that the reasons which renders Einstein's exterior gravitation so effective for the characterization of the *stability* of the planetary orbits and other exterior features are exactly due to the null value of its torsion. The same reasons are then at the origin for the *inability* of the theory to represent the *instability* of the interior orbits (see Fig. 1 of Sect. 10).

In turn, these results necessarily lead Santilli to *two different, but compatible theories: one for the exterior gravitational problem with null torsion, and one for the interior gravitational problem with null isotorsion but non-null*

torsion , as outlined in the next section.

I.12. ISOGENERAL RELATIVITY

The final, and perhaps most significant, geometric discoveries which permitted Santilli (1988d), (1991b, d) the achievement of a geometrically consistent generalization of Einstein's gravitation for the interior problem, consisted of the isotopies of conventional *parallel transport* and *geodesic* , which he called *isoparallel transport* and *isogeodesic* .

Since the times of Galileo Galilei and his experiments at the Pisa tower (1609), we know that *the free fall of a body in Earth's gravitational field is geodesic only in the absence of resistive forces due to our atmosphere.* It is therefore well know that the trajectory of a test particle within a physical medium is not geodesic, owing to the deviations caused by the forces between the body and the medium, as illustrated, say, by a satellite of irregular shape during re-entry in Earth's atmosphere. Moreover, it is also well known since Lagrange's and Hamilton's times (Sect. 1) that the forces between the body and the medium are of nonlinear, nonlocal-integral and nonlagrangian-nonhamiltonian type, that is, of a type outside the representational capabilities of the conventional, local-differential, Riemannian geometry. A fully similar situation occurs for parallel transport.

Thus, not only the conventional notions of geodesic and parallel transport, but the Riemannian geometry itself is inapplicable to the motion of an extended test particle within a physical medium.

After the the identification of a nonlocal-nonlagrangian generalization of the Riemannian geometry reviewed in the preceding section, Santilli's was in a position to achieve the generalization of parallel transport and geodesic for motion of extended particles within physical media, most remarkably, in such a way to preserve the original geometric axioms of the conventional quantities (see later on Fig. 2 in this section).

Santilli's isoparallel transport and isogeodesic are crucial for a true understanding of his isotopic relativities for the interior problem. In fact, the new relativities are based on the preservation of the axioms of the conventional ones. An understanding of the relativity laws in the transition from motion in vacuum to motion within physical media therefore requires the prior understanding of Santilli's preservation of the axiomatic structure of geodesic motion and the generalization instead of the underlying carrier space.

Finally Santilli's isoparallel transport and isogeodesic are a prerequisite of

another crucial aspect of the isotopic relativities, their reconstruction of exact space-time symmetries, such as the rotational symmetry O(3), the Galilei symmetry G(3.1) and the Poincaré symmetry P(3.1), when believed to be conventionally broken.

A typical example is Santilli's reconstruction of the exact *rotational* symmetry O(3) for all infinitely possible *deformations of the sphere*. In fact, the understanding on how the rotational symmetry can be exact, say, for the ellipsoids $x_1 b_1^2 x_1 + x_2 b_2^2 x_2 + x_3 b_3^2 x_3$ = inv. requires the prior knowledge that the geodesic character of the O(3) orbits on a sphere is preserved in the transition to the corresponding orbits on a hyperboloid, provided that one performs the transition from the conventional Euclidean space $E(r,\delta,\Re)$ to Santilli's isospace $\hat{E}(r,\hat{\delta},\hat{\Re})$.

These are the reasons why we have presented in this volume the geometrical foundations of Santilli's *isogravitation* as a necessary condition for the true understanding of his simpler isospecial and isogalilean relativities.

To begin our review, let $R(x,g,\Re)$ be a conventional n-dimensional Riemannian space. Under sufficient smoothness and regularity conditions hereon assumed, a vector field X^i on $R(x,g,\Re)$ is said to be *parallel along a curve* C if it satisfies the differential equation along C (see Lovelock and Rund (*loc. cit.*))

$$DX^i = X^i{}_{|s}\, dx^s = \left(\frac{\partial X^i}{\partial x^s} + \Gamma 2^{\,i}_{r\,s} X^r \right) dx^s = 0, \tag{12.1}$$

where $\Gamma 2_r{}^i{}_s$ is a symmetric connection. Then, by recalling the notions of isodifferential of Sect. 11.11, we have the following

DEFINITION 12.1 (Santilli (1988d), (1991b, d)): An isovector field X^i on an n-dimensional isoriemannian space $\hat{R}(x,\hat{g},\hat{\Re})$ is said to be "isoparallel" along a curve C on $\hat{R}(x,\hat{g},\hat{\Re})$, iff it verified the isotopic equations along C

$$\hat{D}X^i = X^i{}_{\hat{|}r}\, T^r{}_s(x, \dot{x}, ..)\, \hat{d}x^s =$$

$$= \left[\frac{\partial X^i}{\partial x^s} + \hat{\Gamma} 2^{\,i}_{r\,s}\, T^r{}_t(x, \dot{x}, ..)\, X^t \right] T^s{}_p(x, \dot{x}, ..)\, \hat{d}x^p = 0, \tag{12.2}$$

where $\hat{\Gamma} 2^{\,i}_{r\,s}$ is the symmetric isoconnection and the T's are the isotopic elements.

The identity of axioms (12.1) and (12.2) at the abstract level is evident, again, because of the loss of all distinction between the right, modular, associative product, say Xx, and its isotopic generalization X*x = XTx.

To understand the physical differences between the above two definitions,

let us introduce an independent (invariant) parameter s, such that the isovector field $\dot{x} = \partial x/\partial s$ is tangent to C, and let $X^i = X^i(s)$. Consider the curve C at a point P(1) for $s = s_1$ and let $X^i(1)$ be the corresponding value of the isovector field X^i at P(1).

Consider now the transition from P(1) to P(2), i.e., from s_1 to $s_1 + \partial s$. The corresponding transported value of the isovector field $X^i(2) = X^i(1) + \partial X^\mu$ is said to occur under an *isoparallel displacement* on $\hat{R}(x,\hat{g},\hat{\Re})$ in accordance with Definition 12.1, iff

$$\partial X^i = \frac{\partial X^i}{\partial x^r} T^r_{\ s}\, \partial x^s = -\hat{\Gamma}^{2\ i}_{\ rs}\, T^r_{\ p}\, X^p\, T^s_{\ q}\, \partial x^q. \tag{12.3}$$

The iteration of the process up to a finite displacement is equivalent to the solution of the differential equation

$$\frac{\partial X^i}{\partial s} = \frac{\partial X^i}{\partial x^r} T^r_{\ s} \frac{\partial x^s}{\partial s} = -\hat{\Gamma}^{2\ i}_{\ rs}\, T^r_{\ p}\, X^p\, T^s_{\ q} \frac{\partial x^q}{\partial s}, \tag{12.4}$$

By integrating the above expression in the finite interval (s_1, s_2), one reaches the following

LEMMA 12.1 (Santilli (loc. cit.)): The isoparallel transport of an isovector field $X^i(s)$ on an n-dimensional isoriemannian manifold $\hat{R}(x,\hat{g},\hat{\Re})$ from the point s_1 to a point s_2 on a curve C verifies the isotopic laws

$$\hat{X}^i(2) = \hat{X}^i(1) - \int_1^2 \hat{\Gamma}^{2\ i}_{\ rs}(x, \dot{x}, ..)\, T^r_{\ p}(x, \dot{x}, ..)\, X^p(x)\, T^s_{\ q}(x, \dot{x}, ..)\, \dot{x}^q\, \partial s \tag{12.5}$$

where

$$\hat{X}^i(2) - \hat{X}^i(1) = \int_1^2 \partial X^i = \int_1^2 \frac{\partial X^i}{\partial x^p} T^p_{\ q} \frac{\partial x^q}{\partial s} \partial s. \tag{12.6}$$

The physical implications are pointed out by the fact that the isotransported isovector does not start at the value $X^i(1)$, but at the *modified value* $\hat{X}^i(1)$ characterized by Eq.s (12.6). Additional evident modifications are characterized by the isotopic connection $\hat{\Gamma}^2{}_r{}^i{}_s$ and the two isotopic elements T of the r.h.s. of Eq.s (12.5).

These departures from the conventional definition can be better understood

in a *flat isospace*, via the following evident

COROLLARY 12.1.1(loc. cit.): In a flat isospace, such as the isominkowski space $\hat{M}(x,\hat{\eta},\hat{\mathfrak{R}})$ in (3.1)-space-time dimensions, or the isoeuclidean space $\hat{E}(r,\hat{\delta},\hat{\mathfrak{R}})$ in 3-dimensional space, the conventional notion of parallelism no longer holds, in favor of the following flat isoparallelism

$$\hat{\Gamma}^{2}{}^{i}{}_{rs} = 0, \tag{12.7a}$$

$$\hat{x}^{i}(2) - \hat{x}^{i}(1) = \int_{1}^{2} \hat{d}x^{i} = \int_{1}^{2} \frac{\partial x^{i}}{\partial x^{p}} T^{p}{}_{q} \frac{\hat{d}x^{q}}{\hat{d}s} \hat{d}s. \tag{12.7b}$$

Consider, as an illustration, a straight line C in conventional Euclidean space $\mathfrak{R}_t \times E(r,\delta,\mathfrak{R})$, with only two space-components. Then a vector $\bar{R}(1)$ at $s = t_1$ is transported in a parallel way to $\bar{R}(2)$ at $s = t_2$ by keeping unchanged the characteristic angles with the reference axis, i.e.,

$$R^{k}(2) - R^{k}(1) = \int_{1}^{2} \left(\frac{\partial R^{k}(r)}{\partial x^{1}} dx^{1} + \frac{\partial R^{k}(r)}{\partial x^{2}} dx^{2}\right). \tag{12.8}$$

Under isotopy, the situation is no longer that trivial. In fact, assume the simple diagonal isotopy

$$T = \mathrm{diag.}\,(b_1{}^2(r), b_2{}^2(r)) > 0. \tag{12.9}$$

Then Eq.s (12.8) are lifted into the form

$$\hat{R}^{k}(2) - \hat{R}^{k}(1) = \int_{1}^{2} \left(\frac{\partial R^{k}(r)}{\partial r^{1}} b_1{}^2(r)\, \hat{d}r^{1} + \frac{\partial R_{k}(r)}{\partial r^{2}} b_2{}^2(r)\, \hat{d}r^{2}\right. \tag{12.10}$$

In figurative terms, a given straight and rigid arrow in 3-space is, first, twisted under isotopy, and then transported in an isoparallel way, that is, in such a way that the isotopic (rather than the conventional) characteristic angles with the reference axis are preserved (see also the example of isorotation (Santilli (1991d), Chapter III in particular).

It appears that this is exactly the physical behavior of parallel transport within physical media. In fact, one can imagine the rigid arrow as representing a rocket under parallel transport along a straight line toward the center of Earth.

During the motion in the empty space of exterior dynamics, the orientation of the rocket in space remains evidently the same (conventional parallel transport in flat Euclidean space). However, the moment the rocket penetrates Earth's atmosphere, its orientation in space is evidently changed depending on the local conditions (shape of rocket, its density, etc.), even though we can still assume that the center of mass of the rocket keeps moving in a straight line. These are exactly the physical conditions represented by Santilli's isoparallel transport in flat isospaces, Eq.s (12.10).

The *irreducibility* of the notion of isoparallel transport to the conventional notion can be illustrated even in the case of null curvature. In fact, consider for simplicity the isominkowski-space $M(x,\hat{\eta},\hat{\mathfrak{R}})$ with local coordinates $x = (x^\mu)$, $\mu = 1, 2, 3, 4$, and constant diagonal isotopy

$$\hat{\eta} = T\eta, \quad T = \mathrm{diag.}(b_1^2, b_2^2, b_3^2, b_4^2) > 0. \qquad (12.11)$$

and introduce the redefinitions $\hat{x}^\mu = b_\mu^{\ 2} x^\mu$ (no sum), $X^\mu(x(\hat{x})) = \hat{X}^\mu(\hat{x})$.

Then Eq.s (12.7b) become

$$\int_1^2 [\frac{\partial X^\mu(x)}{\partial x^\alpha} b_\alpha^{\ 2} \partial x^\alpha \equiv \int_1^2 \frac{\partial \hat{X}^\mu(\hat{x})}{\partial \hat{x}^\alpha} b_\alpha^{\ 2} \partial \hat{x}^\alpha, \qquad (12.12)$$

namely, the isotopy persists even under the simplest possible constant isotopy (12.11), thus confirming the achievement of a novel geometrical notion.

By submitting the conventional treatment (Sect. 3.7 of Lovelock and Rund (*loc. cit.*)) to isotopies, Santilli then identified the *integrability conditions for the existence of isoparallelism*. By performing partial derivatives of Eq.s (12.7) with respect to x^t and then interchanging symbols, he obtains

$$\frac{\partial X^i}{\partial x^s \partial x^t} = - \frac{\partial \hat{\Gamma}^{2\ i}_{\ rs}}{\partial x^t} T^r_{\ p} X^p + \hat{\Gamma}^{\ i}_{rs} T^r_{\ q} \hat{\Gamma}^{2\ q}_{\ m\ t} T^m_{\ n} X^n +$$

$$+ \hat{\Gamma}^{2\ i}_{\ rs} \frac{\partial T^r_{\ p}}{\partial x^t} X^p = - \frac{\partial X^i}{\partial x^t \partial x^s} = \frac{\partial \hat{\Gamma}^{2\ i}_{\ rt}}{\partial x^t} T^r_{\ p} X^p +$$

$$+ \hat{\Gamma}^{2\ i}_{\ rt} T^r_{\ p} \hat{\Gamma}^{2\ p}_{\ ms} T^m_{\ n} X^n + \hat{\Gamma}^{2\ i}_{\ rt} \frac{\partial T^r_{\ p}}{\partial x^s} X^p \qquad (12.13)$$

from which the following property holds.

LEMMA 12.2 (Santilli (loc. cit.)): Necessary and sufficient conditions for the

existence of an isoparallel transport of an isovector X^i on an n-dimensional isoriemannian space $\hat{R}(x,\hat{g},\hat{\mathfrak{R}})$ are that all the following equations are identically verified

$$\hat{R}_{l\,hk}^{\;i}\, T^l_s X^s = 0, \tag{12.14}$$

where $\hat{R}_{r\,pq}^{\;i}$ is the isocurvature tensor (Sect. 11.11, i.e.

$$\begin{aligned} \hat{R}_{l\,hk}^{\;i} = & \frac{\partial \hat{\Gamma}^{2\,i}_{l\,h}}{\partial x^k} - \frac{\partial \hat{\Gamma}^{2\,i}_{l\,k}}{\partial x^h} + \\ & + \hat{\Gamma}^{2\,i}_{m\,k} T^m_{\;r} \hat{\Gamma}^{2\,r}_{l\,h} - \hat{\Gamma}^{2\,i}_{m\,h} T^m_{\;r} \hat{\Gamma}^{2\,r}_{l\,k} + \\ & + \hat{\Gamma}^{2\,i}_{r\,h} \frac{\partial T^r_{\;s}}{\partial x^k} 1^s_{\;l} - \hat{\Gamma}^{2\,i}_{r\,k} \frac{\partial T^r_{\;s}}{\partial x^h} 1^s_{\;l} \end{aligned} \tag{12.15}$$

The re-emergence of the isocurvature tensor as part of the integrability conditions of isoparallel transport, can then be considered as a confirmation of Santilli's achievement of a novel mathematical notion.

We now pass to the review of *Santilli's isogeodesics*. Let s be an invariant parameter and consider the tangent $\dot{x}^i = \hat{d}x^i/\hat{d}s$ of the curve C on an n-dimensional isoriemannian space $\hat{R}(x,\hat{g},\hat{\mathfrak{R}})$. Its absolute isodifferential is given by

$$\hat{D}\dot{x}^i = \hat{d}\dot{x}^i + \hat{\Gamma}^{2\,i}_{r\,s} T^r_{\;p} \dot{x}^p T^s_{\;q} \hat{d}x^q. \tag{12.16}$$

In accordance with Definition 12.1, $\hat{D}\dot{x}^i$ remains isoparallel along C iff

$$\hat{D}\dot{x}^i = 0. \tag{12.17}$$

We can therefore introduce the following

DEFINITION 12.2 (Santilli (loc. cit.)): The "isogeodesics" of an n-dimensional isoriemannian manifold $\hat{R}(x,\hat{g},\hat{\mathfrak{R}})$ are the solutions of the differential equations

$$\frac{\hat{d}^2 x^i}{\hat{d}s^2} + \hat{\Gamma}^{2\,i}_{r\,s}(x,\dot{x},\ddot{x}..)\, T^r_{\;p}(x,\dot{x},\ddot{x}..)\, \frac{\hat{d}x^p}{ds}\, T^s_{\;q}(x,\dot{x},\ddot{x}..)\, \frac{\hat{d}x^q}{ds} = 0\,.$$

$$(12.18)$$

By recalling that $\hat{d}s = ds$, it is easy to see that *the isogeodesics of flat isospaces remain the straight line (i.e., linear functions of s), while those of curved isospaces remain curves.*

It is a simple but instructive exercise to prove the following

LEMMA 12..3 (loc. cit.): The isogeodesics of an n-dimensional isoriemannian manifold $\hat{R}(x,\hat{g},\hat{\mathfrak{R}})$ are the curves verifying the variational principle .

$$\delta\int \hat{d}s^2 = \delta\int (\hat{g}_{ij}(x, \dot{x}, \ddot{x}..)\, \hat{d}x^i\, \hat{d}x^j)^{\frac{1}{2}} = 0. \tag{12.19}$$

As indicated earlier, the notions of isoparallel transport and isogeodesic have a truly fundamental role in Santilli's geometrization of physical media. Additional comments are presented in Figure 2 below.

We are now finally in a position to briefly outline Santilli's isotopic generalizations of Einstein's relativity, which he submitted under the name of *isogravitation* (Santilli ((1988d), (1991b)), for the interior gravitational problem, i.e., the description of gravitation in the interior of the minimal surface S° encompassing all matter of the celestial body considered. The corresponding coverings of Einstein's special relativity and of the Galilei's relativity are evidently particular cases for null isocurvature.

The generalization was first studied by Gasperini (1984a, b, c) who constructed the first gravitational theory with a local Lorentz-isotopic structure following Santilli (1978a), (1982a), (1983a). Gasperini, however, formulated his studies everywhere in space-time, thus reaching rather severe restrictions on the admissible theories from existing gravitational experiments. Also, Gasperini formulated his gravitational theory in a *conventional* Riemannian space.

The isogravitation outlined below, first of all, restrict the isotopies to the interior problem only, by therefore eliminating any restriction from exterior experiments on the magnitude of the interior isotopy. Secondly, Gasperini's studies themselves have been generalized by Santilli by formulating them in his isoriemannian spaces. In this way, Santilli regains the geodesic and torsionless character of Einstein's gravitation, but at a higher geometric level.

The physical motivations for the need of a suitable generalization of Einstein's gravitation for the interior problem are beyond any credible doubt. The general relativity was specifically conceived (see, Einstein (1916)) for the exterior problem, e.g, for the decription of the planetary motion in our Solar system, which

is well established to be local-differential. The Riemannian geometry is then *exactly* applicable under these exterior conditions.

SANTILLI'S GEODESIC DESCRIPTION OF MOTION WITHIN PHYSICAL MEDIA

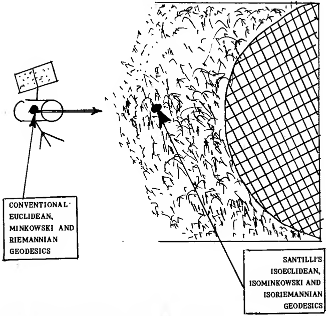

FIGURE 2: The birth of the notion of geodesic motion can be seen in Galilei's historical conception of uniform motion in vacuum, i.e., by the celebrated *Galilei's boosts*

$$r'_k = r_k + t^\circ v^\circ_k, \qquad p'_k = p_k + m v^\circ_k \tag{a}$$

As well known, Galilei established the above law *by ignoring the friction due to the air. Santilli's (1988d), (1991b, d) equally historical discoveries essentially consist in the*

achievement of a geodesic characterization of the motion of free objects within physical media in such a way to preserve the original axioms of the free motion in vacuum.

The fundamental tool in this achievement is provided by the isospaces. In fact, *Santilli represents the transition from motion in vacuum to motion within a physical medium via the transition from conventional Euclidean, Minkowskian or Riemannian space to his corresponding isoeuclidean, isominkowskian and isoriemannian spaces, respectively.* By recalling that the conventional spaces provide a geometrization of the vacuum (empty space), one can then see that *Santilli's isospaces provide a geometrization of physical media.*

Consider, as an example, the conventional Euclidean space $E(r,\delta,\mathfrak{R})$ of Galilei's boosts. In this case the metric $\delta = \mathrm{diag.}\ (1, 1, 1)$ represents the *homogeneity and isotropy of empty space.* Consider now Santilli's treatment. Then his isoeuclidean space $\hat{E}(r,\hat{\delta},\hat{\mathfrak{R}})$ provides a geometrization of the medium because the isometric $\hat{\delta} = T\delta$, $T = T^\dagger > 0$, represents precisely the *inhomogeneity* of the medium (e.g., due to its variation of density with height) as well as its *anisotropy* (e.g., because of a possible angular momentum of the medium which evidently creates a privileged direction in the medium itself).

In particular, the scripture $\hat{\delta} = T\delta$ stands to indicate that the underlying space, represented by δ, remains perfectly homogeneous and isotropic in Santilli's theories, while the isotopic element T represents the mutation of its geometrical characteristics caused by a physical medium. Since T is nonsingular and Hermitean, it can always be diagonalized to a form of the type

$$T = \mathrm{diag.}\ (B_1^{\,2}, B_2^{\,2}, B_3^{\,2}), \tag{b}$$

where the B's, called *characteristics B-quantities of the medium*, are generally nonlinear and nonlocal functions on all variables, $B_k = B_k(t, r, p, \dot{p}, \ldots)$. They represent precisely the interactions between the test body and the medium (see the isoanalytic representations of Sect. 7).

The above explicit functional form of the characteristic B-quantities is needed for the local behavior of a test particle within the medium consider. For the case of a global behavior, such as for the propagation of light throughout our entire atmosphere, the B-quantities can be averaged to constants, $<|B_k|> = b_k$ = const. (see Santilli (1988a), (1991d) for details and applications).

Consider now *Santilli's isoboosts* on $\hat{E}(r,\hat{\delta},\hat{\mathfrak{R}})$ (see Sect. 15 for their derivation)

$$\begin{aligned} r'_k &= r_k + t^\circ v^\circ \tilde{B}_k^{-2}(t, r, p, \ldots), \\ p'_k &= p_k + m v^\circ \tilde{B}_k^{-2}(t, r, p, \ldots), \end{aligned} \tag{c}$$

where the $\tilde{B}$'s are certain functions of the B's given in Theorem 15.1.

It is then evident that *Santilli's isoboosts (c) can represent the deviations from motion in vacuum caused by the medium*. Note in particular that the motion is no longer necessarily uniform, because it can be either decelerated, or accelerated. In fact, when the test particle penetrates a physical medium, it normally decelerates due to the drag force, but it can also accelerate for media in highly dynamical conditions.

Our objective here is to review the reasons why isoboosts (c), first of all, constitute an geodesic motion on isospace $\hat{E}(r,\hat{\delta},\hat{\Re})$ and, second, they preserve the axioms of the original Galilei's boosts (Santilli (*loc. cit.*)).

The geodesic character is readily proved (under predictable restrictions on the $\hat{B}$-functions). In fact, the geodesics in $E(r,\delta,\Re)$ are characterized by the trivial expression d^2r_k / $dt^2 = 0$ whose only possible solutions are given precisely by the Galilei boosts (a). In the transition to isospace $\hat{E}(r,\hat{\delta},\hat{\Re})$, the geodesic equations are given instead by $\hat{d}^2r_k \;/\; \hat{d}t^2 = 0$ which, by recalling property (9.47), do indeed admit non trivial solutions of form (c). Fully similar cases occur in Santilli's isospecial and isogeneral relativities.

We remain with showing that the axiomatic structure of Galilei's boosts is preserved by Santilli's generalization. This is readily seen via the use of the Lie-Santilli theory. Consider the one-dimensional Lie group $T(v^\circ)$ representing Galilei's boosts. Then, expressions (a) are given by the familiar forms (see, e.g., Sudarshan and Mukunda (1974))

$$T(v^\circ)\, r_k \;=\; r_k \;+\; t^\circ\, v^\circ_k, \quad T(v^\circ)\, p_k \;=\; p_k \;+\; m\, v^\circ_k \tag{d}$$

Thus, from an axiomatic viewpoint, the Galilei boosts are characterized by the conventional, modular, associative action $T(v^\circ)\, r_k$ and $T(v^\circ)\, p_k$.

In the transition to Santilli's isogalilean relativity (Sect. 15), the isoboosts are represented instead by the one-parameter Lie-isotopic group $\hat{T}(v^\circ)$ with the same generator and parameter of $T(v^\circ)$, but now expanded in the isoenvelope (Sect. 6). Expressions (c) are then explicitly given by the now familiar modular, isoassociative actions

$$\hat{T}(v^\circ){*}r_k \;=\; r_k \;+\; t^\circ\, v^\circ\, \hat{B}_k^{-2}, \quad \hat{T}(v^\circ){*}p_k \;=\; p_k \;+\; m\, v^\circ\, \hat{B}_k^{-2} \tag{e}$$

The preservation of the geometric axioms of Galilei's relativity by Santilli's covering isorelativity is then ensured by the fact that all distinctions between the conventional modular actions (d) and their isotopic generalization (e) cease to exist at the abstract, realization-free level.

The preservation of the original axiomatic structure is then confirmed by the property that, for two sequential successions of Galilei's boosts we have the familiar group composition law

$$T(v^\circ)\; T(v^{\circ\prime}) \;=\; T(v^\circ + v^{\circ\prime}) \tag{f}$$

while for Santilli's isoboosts we have the covering, fully equivalent composition law of

isotopic groups

$$\hat{T}(v^\circ) * \hat{T}(v^{\circ\prime}) = \hat{T}(v^\circ + v^{\circ\prime}) \tag{g}$$

In conclusions, *Santilli's has achieved a generalization of the algebraic, analytic and geometric structures of Galilei's, Einstein's special and Einstein's general relativities for the most general known physical conditions: motion of extended-deformable particles within physical media experiencing conventional, action-at-a-distance, potential forces, as well as contact, short range, nonlinear, nonlocal and nonlagrangian-nonhamiltonian forces due to the medium itself*.

In particular, *the generalizations are such to preserve the axiomatic structure of the original relativities, to such an extent that any distinction between the conventional and Santilli's relativities cease to exist at the abstract, realization-free level by conception*.

In the transition to the interior problem, the physical conditions are fundamentally different, to such an extent to render inapplicable the Riemannian geometry itself, let alone Einstein's conception of gravitation.

In fact, as recalled in Sect. 1, according to incontrovertible evidence, the core of of a star undergoing gravitational collapse is composed of the wavepackets of particles in conditions of total mutual immersion, and their compression in very large numbers in a small region of space. This results in nonlinear, nonlocal, and nonlagrangian-nonhamiltonian internal interactions of type (1.1) which simply require geometries structurally more general than the Riemannian one.

The reader who is familiar with the isotopic geometries can now see that the isotopic liftings of Einstein's gravitation permit the transition precisely from the exterior problem for a given gravitational mass, to the infinite possibilities of interior gravitational conditions for each given total mass, evidently caused by the infinitely possible interior media.

Let us begin by recalling the three basic representation spaces used in the conventional approach to gravity:

1) The carrier space of Galilean exterior mechanics, the Kronecker product of *Euclidean spaces* for conservative trajectories

$$\Re_t \times E(r, \delta, \Re) : \quad \delta = \mathrm{diag.}\,(1, 1, 1), \tag{12.20a}$$

$$r = (r^i), \quad r^2 = r^t \delta r = r^i \delta_{ij} r^j = r_1{}^2 + r_2{}^2 + r_3{}^2, \tag{12.20b}$$

$$i = 1, 2, 3 \quad (= x, y, z)$$

over the reals $\Re$;

2) The carrier space of relativistic exterior mechanics, the familiar

Minkowski space for exterior trajectories

$$M(x,\eta,\Re)\ :\ \eta = \text{diag.}\,(1,1,1,-1), \tag{12.21a}$$

$$x = (x^\mu) = (r, x^4), \quad x^4 = c_o t, \quad r \in E(r,\delta,\Re)$$

$$x^2 = x^t \eta x = x^\mu\,\eta_{\mu\nu}\,x^\nu = x_1{}^2 + x_2{}^2 + x_3{}^2 - x_4{}^2 = R^2\,; \tag{12.21b}$$

$$\mu, \nu = 1, 2, 3, 4, \quad R \in \Re,$$

where c_o represents hereon the *speed of light in vacuum;* and

3) The carrier space of Einstein's exterior gravitation, the *Riemannian space* in the conventional (3.1)-space-time with a sym-metric connection and null torsion

$$R(x,g,\Re)\ : \qquad g = g(x) = (g_{\mu\nu}) = (g_{\nu\mu}), \tag{12.22a}$$

$$x = (x^\mu) = (r, x^4); \qquad x^2 = x^t g(x)\, x = x^\mu g_{\mu\nu} x^\nu = R^2, \tag{12.22b}$$

$$\Gamma^1{}_{\mu\rho\nu} = \tfrac{1}{2}\Big(\frac{\partial g_{\mu\rho}}{\partial x^\mu} + \frac{\partial g_{\rho\nu}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\rho} \Big), \tag{12.22c}$$

$$T_\mu{}^\rho{}_\nu = \Gamma^2{}_\mu{}^\rho{}_\nu - \Gamma^2{}_\nu{}^\rho{}_\mu \equiv 0. \tag{12.22d}$$

As well known, the largest groups of linear and local-differential isometries of space (12.20) is the *Galilei symmetry* $G_\delta(3.1)$. The largest group of linear and local isometries of the Minkowski space (12.21) is the *Poincaré symmetry* $P_\eta(3.1)$. The largest group of isometries of the Riemannian space (12.22) is known only locally, i.e., in the neighborhood of a point and it is given also by $P_\eta(3.1)$. The global symmetry of spaces (12.22) was identified, apparently for the first time, by Santilli ((1988d), (1991b, d)).

The first contribution by Santilli's isotopic formulations is provided for *conventional* gravitational theories, and can be formulated as follows.

THEOREM 12.1 (Santilli (loc. cit.)): The largest possible nonlinear but local-differential group of isometries of the conventional Riemannian spaces (12.22) with metric g(x) are Santilli's isopoincaré symmetries $\hat{P}_g(3.1)$ with isounits $\hat{1} = T^{-1}$, $g = \hat{T}$. For the case of Einsteinian theories of gravity, $T > 0$ and all isosymmetries $\hat{P}_g(3.1)$ are locally isomorphic to the conventional Poincaré symmetry P_η (3.1) (see Sect. 13 for details).

Note that the above theorem is not purely formal, but permits the computation of the explicit symmetry transformations in their finite form from the sole knowledge of each given metric g(x), say, the Schwarzschild's metric.

Also, recall the conventional nonrelativistic limit

$$P_\eta(3.1)\big|_{c_o/R \Rightarrow 0} \Rightarrow G_\delta(3.1). \tag{12.23}$$

We can then say that the Poincaré-isotopic symmetries $\hat{P}_g(3.1)$ admit a nonrelativistic limit into a symmetry $\hat{G}_{\tilde{\delta}}(3.1)$ which are also locally isomorphic to the Galilei symmetry $G_\delta(3.1)$ whenever $\tilde{\delta} > 0$ (as it is the case for Einsteinian gravitational theories),

$$\hat{P}_g(3.1)\big|_{c_o/R \Rightarrow 0} \Rightarrow \hat{G}_{\tilde{\delta}}(3.1) \approx G_\delta(.3.1). \tag{12.24}$$

Finally, recall that $\hat{P}_g(3.1)$ admit the conventional symmetry $P_\eta(3.1)$ as a local relativistic symmetry, and $\hat{G}_\delta(3.1)$ as a local nonrelativistic symmetry (see Appendix A of Santilli (1988c)).

This essentially summarizes the geometrical structure of the local and global symmetries of Einstein's exterior gravitation.

Santilli's objective was then to reach an infinite class of *symmetry-preserving isotopies* of Einstein's gravitation for the interior dynamical problem, and interpret the isotopies, as now familiar, as representing the transition from motion in vacuum to motion within a physical medium.

Note that the *axiom-preserving isotopies* cannot be introduced for Einstein's gravitation because of the technical difficulties caused by the lack of axiomatic completeness of Einstein's tensor discussed in Sect. 11, although they can be submitted for geometrically complete theories (see below).

To identify the structure of the isotopies of Einstein's gravitation, let us review the three basic isospaces of the analysis:

1') Santilli's carrier space of the nonrelativistic[20] interior mechanics, the (flat) *isoeuclidean spaces* for nonlinear, nonlocal and nonlagrangian trajectories

$$\hat{\mathfrak{R}}_t \times \hat{E}(x,\hat{\delta},\hat{\mathfrak{R}}): \quad \hat{\delta} = \hat{\delta}(r, \dot{r}, \ddot{r}, ...) = T_{\hat{\delta}}(r, \dot{r}, \ddot{r}, ...)\,\delta = (\hat{\delta}_{ij}) = (\hat{\delta}_{ji}), \tag{12.25a}$$

[20] Santilli generally avoids the term "Newtonian" for the interior dynamical problem, and uses instead terms such as "nonrelativistic", "isogalilean" or "isonewtonian", because of the presence of forces in Eq.s (1.1) that are acceleration dependent and, as such, not considered "Newtonian" on strict grounds. It should be indicated that acceleration-dependent forces possess rather intriguing and mostly unexplored implications (see, e.g., Assis (1990), Graneau (1990)).

$$r^{\hat{2}} = r^t \hat{\delta}\, r = r^i \hat{\delta}_{ij}(r, \dot{r}, \ddot{r}, ..)\, r^j, \tag{12.25b}$$

$$\hat{\mathfrak{R}} = \mathfrak{R}\, \hat{1}_{\hat{\delta}}, \quad \hat{1}_{\hat{\delta}} = T_{\hat{\delta}}^{-1}, \quad \hat{\mathfrak{R}}_t = \mathfrak{R}_t \hat{1}_t; \quad \hat{1}_{\hat{\delta}} > 0, \quad \hat{1}_t > 0; \tag{12.25d}$$

2) Santilli's carrier space of relativistic interior mechanics, the (flat) *isominkowski spaces* for the relativistic description of nonlinear, nonlocal, nonlagrangian and nonlorentzian trajectories

$$\hat{M}^I(x, \hat{\eta}, \hat{\mathfrak{R}}): \; \hat{\eta} = \hat{\eta}(x, \dot{x}, ...) = T_{\eta}(x, \dot{x}, ...)\, \eta = (\hat{\eta}_{\mu\nu}) = (\eta_{\nu\mu}), \tag{12.26a}$$

$$x^{\hat{2}} = x^t \hat{\eta}(x, \dot{x}, \ddot{x}, ..)\, x = x^{\mu} \hat{\eta}(x, \dot{x}, \ddot{x}, ..)\, x^{\nu} = R^2, \tag{12.26b}$$

$$\hat{\mathfrak{R}} = \mathfrak{R}\, \hat{1}_{\hat{\eta}}, \quad \hat{1}_{\hat{\eta}} = T_{\hat{\eta}}^{-1} > 0, \quad x, \eta \in M(x, \eta, \mathfrak{R}), \quad R \in \mathfrak{R}. \tag{12.26c}$$

Also called by Santilli *isominkowski space of class*, $\hat{M}^I(x, \hat{\eta}, \hat{\mathfrak{R}})$.

Finally, we introduce

3') Santilli's carrier space of gravitational interior theories, the *isoriemannian spaces* in (3.1)-dimension with a symmetric isoconnection and a null isotorsion in the geometrical space, but a non-null torsion in the physical space of the observer

$$\hat{R}(c, \hat{g}, \hat{\mathfrak{R}}): \hat{g} = \hat{g}(x, \dot{x}, ...) = T_g(x, \dot{x}, ..T)\, g(x)$$

$$= (\hat{g}_{\mu\nu}) = (\hat{g}_{\nu\mu}) = (T_{\mu}{}^{\sigma} g_{\sigma\nu}), \tag{12.27a}$$

$$x^{\hat{2}} = x^t \hat{g}(x, \dot{x}, ..)\, x = x^{\mu} \hat{g}(x, \dot{x}, ...)\, x^{\nu} = R^2, \tag{12.27b}$$

$$\hat{\Gamma}^1{}_{\mu\rho\nu} = \tfrac{1}{2}\left(\frac{\partial \hat{g}_{\mu\rho}}{\partial x^{\mu}} + \frac{\partial \hat{g}_{\rho\nu}}{\partial x^{\nu}} - \frac{\partial \hat{g}_{\mu\nu}}{\partial x^{\rho}}\right) \tag{12.27c}$$

$$\hat{\tau}_{\mu}{}^{\rho}{}_{\nu} = \hat{\Gamma}^2{}_{\mu}{}^{\rho}{}_{\nu} - \hat{\Gamma}^2{}_{\nu}{}^{\rho}{}_{\mu} \equiv 0, \tag{12.27d}$$

$$\tau_{\mu}{}^{\rho}{}_{\nu} = T_{\mu}{}^{\sigma}\, \hat{\Gamma}^2{}_{\sigma}{}^{\rho}{}_{\nu} - T_{\nu}{}^{\sigma}\, \hat{\Gamma}^2{}_{\sigma}{}^{\rho}{}_{\mu} \neq 0. \tag{12.27e}$$

As proved in Santilli (1988a), the largest possible nonlinear and nonlocal-integral groups of isometries of the isoeuclidean space (12.25) are given by the *Galilei-isotopic symmetries* $\hat{G}_{\hat{s}}(3.1)$. The largest possible nonlinear and nonlocal

groups of isometries of the isominkowski spaces (12.26) are given by the Poincaré–isotopic symmetries $\hat{P}_{\hat{\eta}}(3.1)$. The same methods then lead to the following

THEOREM 12.2 (Santilli (1988d), (1991b, d)): The largest possible nonlinear and nonlocal groups of isometries of the isoriemannian spaces are Santilli's isopoincaré symmetries $\hat{P}_{\hat{g}}(3.1)$ for isometrics $\hat{g} = Tg$ with (nowhere singular and Hermitean) isounits $\hat{1} = T^{-1}$, which result to be locally isomorphic to the conventional Poincaré symmetry $P_{\eta}(3.1)$ for all infinitely possible, positive-definite isotopic elements T_g.

Therefore, the fundamental space-time symmetry of Einstein's gravitation is not lost in the transition to interior gravitational problems, but merely realized in the most general known, nonlinear and nonlocal, isotopic form (see Fig. 3 below and Sect. 13 for more details).

In particular, in Appendix IV.A of Santilli (1991d) it is shown that, in full analogy with property (12.23),

$$\hat{P}_{\hat{\eta}}(3.1)\Big|_{c_o/R\Rightarrow 0} = \hat{G}_{\hat{\delta}}(3.1), \tag{12.28}$$

Similarly, the global symmetry $\hat{P}_{\hat{g}}(3.1)$ admits, locally, the relativistic isosymmetry $\hat{P}_{\hat{\eta}}(3.1)$ and the nonrelativistic isosymmetry $\hat{G}_{\hat{\delta}}(3.1)$. In this way, every major aspect of the conventional theory has been shown to admit an infinite number of corresponding isotopic liftings.

More particularly, isotopic spaces (12.25), (12.26) and (12.27) provide an infinite number of *coverings* of the corresponding conventional spaces (12.20), (12.21) and (12.22), respectively; similarly, isosymmetries $\hat{P}_{\hat{g}}(3.1)$, $\hat{P}_{\hat{\eta}}(3.1)$ and $\hat{G}_{\hat{\delta}}(3.1)$ provide an infinite number of coverings of the corresponding conventional symmetries $P_g(3.1)$, $P_\eta(3.1)$ and $G_\delta(3.1)$.

We are now sufficiently equipped to present the *infinite number of isotopic liftings of Einstein's gravitation for the interior gravitational problem*, called *Einstein–isotopic gravitation*, or *isogravitation* for short, which can be introduced via the following generalized equations on isoriemannian space $\hat{R}(x,\hat{g},\hat{\mathfrak{R}})$ in standard units (Santilli (1988d), (1991b), (1991d)

$$\delta\hat{A} = \delta\int d^4x\, \hat{\Delta}^{\frac{1}{2}}(\hat{R} - 8\pi\hat{M}) =$$

$$= \delta\int d^4x\, \Delta^{\frac{1}{2}}(\hat{g}^{\mu\nu}\hat{g}^{\rho\sigma}\hat{R}_{\mu\nu\rho\sigma} - 8\pi\hat{g}^{\mu\nu}\hat{M}_{\mu\nu}) = 0, \tag{12.29a}$$

$$\hat{g} = T_g g, \qquad T_g > 0, \tag{12.29b}$$

$$T_{g|r>R^\circ} = I = \text{diag.}\,(1,1,1,1), \tag{12.29c}$$

where:

I) Eq. (12.29a) represents the isotopic action on $\hat{R}(x,\hat{g},\hat{\mathfrak{R}})$ (Theorem 11.5), with $\hat{R}$ being the isocurvature isoscalar and $\hat{M}$ the isoscalar of the conventional matter tensor computed and contracted on $\hat{R}(x,\hat{g},\hat{\mathfrak{R}})$. This first condition ensures the achievement of a generalized theory of the interior gravitational problem, with particular reference to the admission of nonlinear, nonlocal, nonlagrangian and non-Newtonian internal trajectories, as well as a direct representation of the inhomogeneity and anisotropy of the interior physical media, as evidently permitted by the isotopic element T_g.

II) Condition (12.29b), is imposed to preserve the topological structure of the exterior treatment $g = T_\eta\eta$, $T_\eta > 0$, also in the interior problem with $\hat{g} = T_g g$, $T_g > 0$, thus allowing the fundamental preservation of the Poincare' symmetry as the universal symmetry for local and global, interior and exterior conditions.

III) Condition (12.29c) implies that the isotopic element T_g acquires the trivial unit value I = diag. (1,1,1,1) everywhere in the exterior problem, by therefore guaranteeing that the isogravitation recover the conventional Einstein's gravitation identically everywhere in the exterior problem.

An inspection of the various metrics then implies that *in the transition from the exterior to the interior problem there is the transition from a local-differential dynamics with (variationally) selfadjoint interactions describing motion in the homogeneous and isotropic vacuum, to a nonlocal-integral dynamics with selfadjoint and nonselfadjoint interactions describing motion within generally inhomogeneous and anisotropic interior physical media.*

In different terms, Einstein's exterior gravitation represents the trajectories of dimensionless test particles in vacuum which, as such, can only have a local-differential geometry with action-at-a-distance dynamics on stable orbits, as well known.

In the transition to our interior problem, we have instead the representation of extended (and therefore deformable) test bodies moving within resistive media which, as such, demand a nonlocal-integral geometry and nonlagrangian dynamics.

The direct representation of interior physical media is evidently ensured by the isometrics $\hat{g}(x, \dot{x}, \ddot{x}, \mu, \tau, n, \ldots)$ which can represent physical notions and events essentially beyond the representational capabilities of Einstein's gravitations, such as:

a) the variation of the density μ of the interior medium with the distance from the center (inhomogeneity);

b) a preferred direction in the interior medium caused by the intrinsic angular momentum of the body (anisotropy);

c) the local variation of the index of refraction n,

d) the local dependence of the speed of light $c = c_o b_4(x, \dot{x}, \mu, \tau, ...)$ on the physical conditions at end (when the medium is transparent);

e) the maximal causal speed which becomes a local quantity dependent on the physical conditions at the point considered;

and numerous additional features typical of interior dynamics that are suitable for direct experimental verification (see next section).

The reader should keep in mind the following chains of isotopies for the exterior and interior problem, respectively,

$$1_t \times \delta \Rightarrow \eta = T(1_t \times \delta) \Rightarrow g = T\eta, \tag{12.30a}$$

$$\hat{1}_t \times \hat{\delta} \Rightarrow \hat{\eta} = T(\hat{1}_t \times \hat{d}) \Rightarrow \hat{g} = \hat{T}\hat{\eta}, \tag{12.30a}$$

which are important to understand Santilli's interpretation of the corresponding chains of relativities as one being the isotopes of the others (see Fig. 3 below).

By recalling that the Lie-isotopic liftings of Lie's symmetries preserve, by construction, the original generators (and parameters), the physical implications of the above results is expressed by the following property directly originating from the Lie-isotopic theory.

THEOREM 12.3 (Santilli (loc. cit.)): All global and local, conventional and isotopic Poincaré symmetries of isogravitation (12.29) admit as generators the same, conventional, total conserved quantities

$$P^\mu = \int M^{\mu 0}\, d^3x, \tag{12.31a}$$

$$J^{\mu\nu} = \int (x^\mu M^{\nu 0} - x^\nu M^{\mu 0})\, d^3x. \tag{12.31b}$$

Note that in conventional presentations of Einstein's gravitation, the above total conservation laws are deduced via rather complex methods, while in Santilli's approach the same conservation laws are directly derived from the global Poincare symmetry.

Moreover, the *conventional* total conservation laws (5.14) occur under a *generalized* interior dynamics as expressed in the following property (see Santilli (1988d) for details).

COROLLARY 12.3.1 (loc. cit.): Isogravitation (12.29) characterizes the gravitational

extension of closed-isolated systems with selfadjoint and nonselfadjoint internal forces without subsidiary constraints, for which the total, conventional, conservation laws are ensured by Santilli's global isopoincaré symmetry.

Next, we review the isotopic form of the principle of equivalence. For this purpose, recall the isonormal coordinates of Sect. 11, under which we have the reduction of the isometric $\hat{g}(x, \dot{x}, \ddot{x},...)$ to the tangent isometric $\hat{\eta}(\dot{x}, \ddot{x},...)$ of the flat isominkowski spaces $\hat{M}^1(x,\hat{\eta},\hat{\mathfrak{R}})$..

The following formulation is then rather natural.

PRINCIPLE OF ISOEQUIVALENCE: Gravitational effects in the absence of the source tensor $M^{\mu\nu}$ on Santilli's isoriemannian spaces $\hat{R}(x,\hat{g},\hat{\mathfrak{R}})$ can be locally made to disappear by transforming the isometric $\hat{g}$ into that $\hat{\eta}$ of the tangent isopoincaré space, or in the neighborhood of an isonormal point $y°$ at which

$$\hat{\Gamma}^{2}{}_{\rho}{}^{\mu}{}_{\sigma}(y^{\circ}) = 0. \tag{12.32}$$

The first formulation of the above principle was reached by Gasperini (1984a, b, c), but expressed in a conventional Riemannian space, with consequential lack of nonlocal internal interactions. The full formulation of the principle (and its name) is due to Santilli (1988d), (1991b, d).

The primary novelty of the isoequivalence over the conventional equivalence in vacuum is that, in the latter case the test particle can be made locally free, while in the former case the test particle remains under the action of the contact nonpotential interactions in the neighborhood of the point considered.

This leads to *"Santilli's (1988c), (1991d) No No-Interaction Theorem"*, which essentially establishes that, while a classical relativistic system of particles moving in vacuum which is invariant under the Poincaré symmetry cannot admit interactions (conventional "No-Interaction Theorem"), the corresponding relativistic systems in interior conditions which is invariant under the isopoincaré symmetry cannot be reduced to a free form.

The isogeodesic character of Santilli's interior trajectories has been discussed in Fig. 2.

In conclusion, it appears that Santilli's isogravitation (12.29) is capable of:

1) Admitting a novel, global, isopoincaré symmetry $\hat{P}_g(3.1)$ for the characterization of the conventional Einstein's gravitation, via the embedding of the curvature in the isounit of the theory;

2) Directly representing the conventional total conservation laws (12.29c) via the generators of the global Poincaré-isotopic symmetry;

3) Not being detectable from the outside, because of the conventional

character of the total conservation laws, as inherent in all closed nonselfadjoint systems;

4) Recovering Einstein's gravitation identically in the exterior problem;

5) Verifying all exterior experiments verified by Einstein's gravitation;

6) Preserving the geometries underlying Einstein's gravitation in the transition to the interior problem, including the affine geometry and the Riemannian geometry, although realized in their most general possible form;

7) Representing the most general possible linear or nonlinear, local or nonlocal, Lagrangian or nonlagrangian, Newtonian or non-Newtonian interior trajectories;

8) Preserving the geodesic character, in the transition from the exterior to the interior problem.

9) Predicting a new series of local and global *interior* phenomena which can be subjected to direct experimental verification, such as the apparent isotopic deviations from the Einsteinian Doppler's redshift for light propagating within inhomogeneous and anisotropic transparent media (see Sect. 13), and other interior effects.

Unfortunately, Einstein's exterior gravitation is afflicted by rather serious problematic aspects of numerous and diversified nature, including:

A) Problems of geometric consistency caused by the incompleteness of Einstein's tensor (Lemma (11.2), the lack of invariance of the contracted Bianchi identity under isotopies, and others;

B) An apparent incompatibility of the sourceless character of Einstein's field equations in vacuum,

$$G_{\mu\nu} = 0, \tag{12.33}$$

with Maxwell's electro-dynamics and the electromagnetic origin of matter, the latter implying the necessary existence of a first-order source $T^{elm}{}_{\mu\nu}$ of electromagnetic nature in the exterior of gravitational masses with null total charge and electromagnetic phenomenology with modified exterior field equations in vacuum (Santilli (1974)

$$G_{\mu\nu} = 8\pi\, T^{elm}{}_{\mu\nu}; \tag{12.34}$$

C) Numerous theoretical and experimental problems caused by the lack of stress-energy tensor $t^{stress}{}_{\mu\nu}$ pointed out by Yilmaz (1958), (1971), (1977), (1979), (1980), (1982), (1989), (1990a, b)) with modified field equations in vacuum

$$G_{\mu\nu} = 8\pi\, t^{stress}_{\mu\nu}\,; \tag{12.35}$$

and combined form

$$G_{\mu\nu} = 8\pi\,(T^{elm}_{\mu\nu} + t^{stress}_{\mu\nu}); \tag{12.36}$$

as well as other problematic aspects which have remained unresolved because of lack of consideration by independent experts in the field.

Regrettably, we cannot review the above aspects to avoid a prohibitive length of this manuscript. At any rate, it is an easy prediction that these aspects will remain fundamentally unresolved until confronted by other experts in gravitation and proved to be either correct or erroneous.

This unfortunate condition of the sector essentially leaves the *exterior* gravitational problem in a state of "suspended animation", without any possibility to reach a true scientific conclusion at this time either in favor of Einstein's exterior equations (12.33) or in favor of its generalization (12.36).

The point to be stressed here is that *the problematic aspects of Einstein's exterior gravitation do not affect Santilli's interior isogravitation (12.29), first of all, because it is a theory for the interior gravitational problem and, as such, necessarily admitting of a number of sources and, secondly, because of the degrees of freedom offered by its isotopic structure.*

Santilli (1974), (1988d), (1991d) complete his gravitational studies with the identification of the most general possible isogravitational theory, with an axiomatically correct structure invariant under isotopies (that based on the "completed Einstein-Santilli tensor" of Sect. 11).

In particular, Santilli called the emerging model a *"theory on the origin of the gravitational field"* because it eliminates the now vexing problem of "unification" of the gravitational and electromagnetic field, by replacing it with their "identification" , except for corrections due to the short range, weak and strong interactions.

In particular, the study offer the possibility of identifying a conceivable physical origin of Yilmaz's stress-energy tensor $T^{stress}_{\mu\nu}$ as being an exterior manifestation of the short range, weak and strong interactions in the structure of matter.

These rather basic additional results by Santilli can be readily seen by considering the case of the π° particle. Recall that any field is an origin of the gravitational field. then, the gravitational mass of the π° is well established by

quantum electrodynamics to be of primary electromagnetic structure.

Specifically Santilli (1974) proved that, even though the total charge and electromagnetic phenomenology of the $\pi°$ is null, the particle possesses in its exterior a first-order source $T^{elm}_{\mu\nu}$ originating from the electromagnetic fields of its charged constituents, which is of such high value, to be of the order of magnitude of the conventional Einsteinian mass source

$$M^{Einstein}_{\mu\nu} \sim T^{elm}_{\mu\nu}. \tag{12.37}$$

But the short range (s.r.), weak and strong interactions in the interior of the $\pi°$ also are a (quantitatively smaller) source of its gravitational field. This yields Santilli's hypothesis on the origin of the gravitational field

$$\hat{M}^{Einstein}_{\mu\nu} \equiv \hat{T}^{elm}_{\mu\nu} + \hat{t}^{sr}_{\mu\nu}. \tag{12.38}$$

where the "hat" indicates that the quantities are of the interior problem.

In the transition from the interior to the exterior problem Santilli then gets from hypothesis (12.38)

$$(\hat{T}^{elm}_{\mu\nu} + \hat{t}^{sr}_{\mu\nu})|_{r > S°} = T^{elm}_{\mu\nu} + t^{stress}_{\mu\nu}. \tag{12.39}$$

In fact, the electromagnetic tensor is traceless and, as such, it cannot be confused with Yilmaz's stress-energy tensor. On the contrary, the short range interactions characterize a tensor possessing all the geometric characteristics of a stress-energy tensor, including its lack of traceless character.

By keeping in mind all the various properties and definitions of Sect. 11 here not repeated for brevity, the above results can be expressed by the following

THEOREM 12.4 (ORIGIN OF THE GRAVITATIONAL FIELD; Santilli (1974), (1988d), (1991d): The most general possible formulation of isogravitation on isoriemannian spaces $R(x,\hat{g},\hat{\Re})$ can be expressed via the following variational principle

$$\delta\hat{A} = \delta\int d^4x\, [\hat{R} - 8\pi(\hat{T}^{elm} + \hat{t}^{sr})]$$

SANTILLI'S GEOMETRIC UNIFICATION OF EXTERIOR AND INTERIOR RELATIVITIES

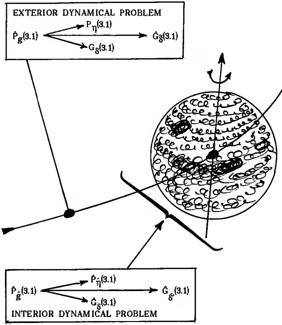

FIGURE 3: A schematic view of Santilli's (1988d), (1991b, d) final geometric unification of Galilei's, Einstein's special and Einstein's general relativities for the exterior problem, as well as their isotopic generalizations for the interior problem, into one, ultimate, abstract symmetry, the isopoincaré symmetry **P[3.1]**, which is realized in isogravitation (12.29) or (12.40) in the following variety of ways of increasing complexity and methodological needs:

A) As the linear and local, conventional symmetry $P_\eta(3.1)$ of the tangent, exterior gravitational problem in normal coordinates;

B) As the global, nonlinear but local isotopic symmetry $\hat{P}_g(3.1)$ of conventional Einstein's exterior gravitation;

C) As the nonlinear and nonlocal isotopic symmetry $\hat{P}_{\hat{\eta}}(3.1)$ for the isonormal coordinates of the interior gravitation;

D) As the global, nonlinear and nonlocal isosymmetry $\hat{P}_{\hat{g}}(3.1)$ for the interior gravitation; and

E) As the Galilean $G_{\delta}(3.1)$ and isogalilean $\hat{G}_{\hat{\delta}}(3.1)$ symmetries under isogroup contractions (see Sect. 14).

The reader can now understand *Santilli's interpretation of Einstein's general relativity as an isotopy of the special relativity.* In fact, the conventional Riemannian space $R(x,g,\Re)$ can be interpreted as an isotope of the Minkowski space (Sect. 3), $\hat{M}(x,g,\hat{\Re})$, $g = T\eta$, $\hat{\Re} = \Re\hat{1}$, $\hat{1} = T^{-1}$.

This discovery as profound geometrical implications because it implies the possibility of reducing both Einstein's special and the general relativity to one single set of abstract axioms. In fact, this property has allowed Santilli to achieve one single global symmetry, the global isopoincaré symmetry $\hat{P}_g(3.1)$ of Einstein's gravitation, which admits the conventional Poincaré symmetry $P_{\eta}(3.1)$ of the special relativity as a particular case.

Along the same lines, one can see that Santilli's isogeneral relativity is an isotope of the isospecial, and both admit one single axiomatic structure characterized by the isopoincaré symmetry $\hat{P}_{\hat{g}}(3.1)$, which admits as a particular case the symmetry $\hat{P}_{\hat{\eta}}(3.1)$ of the isospecial relativity.

The isoriemannian geometry therefore permits the achievement of an ultimate unity of mathematical and physical thought characterized by the multiple infinities of Poincaré-isotopic symmetries which, for positive-definite isotopic elements T, are all locally isomorphic to the conventional one, $\hat{P}_{\hat{g}}(3.1) \approx \hat{P}_{\hat{\eta}}(3.1) \approx P_g(3.1) \approx P_{\eta}(3.1)$ (see next section for details).

But the conventional relativities are a particular case of the isotopic relativities. One reaches in this way Santilli's ultimate geometric unification of this figure.

In this way, Santilli has reached the remarkable synthesis of reducing all possible exterior or interior, relativistic or gravitational, linear or nonlinear, local or nonlocal, Lagrangian or nonlagrangian systems to one, single, unique geometric structure: the isopoincaré symmetry.

$$= \delta\int d^4x\,[\hat{g}^{\mu\nu}\hat{g}^{\rho\sigma}\hat{R}_{\mu\nu\rho\sigma} + 8\pi\hat{g}_{\mu\nu}(\hat{T}^{\mu\nu}{}_{elm} + \hat{t}^{\mu\nu}{}_{sr})] = 0 \qquad (12.40a)$$

$$\hat{g} = T_g g, \qquad T_g > 0, \quad g \in R(x,g,\Re), \qquad (12.40b)$$

$$T_g\big|_{r>R^\circ} = I = \text{diag.}\,(1,1,1,1), \qquad (12.40c)$$

where $\hat{T}^{elm}$ and $\hat{t}^{sr}$ are the electromagnetic and short range fields, respectively, of all individual elementary constituents of matter. Euler–Lagrange equations are then given by

$$\hat{E}^{\mu\nu} = \hat{R}^{\mu\nu} - \tfrac{1}{2}\hat{g}^{\mu\nu}\hat{R} - \tfrac{1}{2}\hat{g}^{\mu\nu}\hat{\theta} - 8\pi(\hat{\tau}^{\mu\nu} + \hat{t}^{\mu\nu}{}_{stress} =$$

$$= \hat{R}^{\mu\nu} - \tfrac{1}{2}\hat{g}^{\mu\nu}\hat{R} - 8\pi(\hat{T}^{\mu\nu}{}_{elm} + \hat{t}^{\mu\nu}{}_{sr}) = 0 \qquad (12.41)$$

whose exterior limit in vacuum for the case of an astrophysical body with null total electromagnetic phenomenology is given by Eq.s (12.36).

Isogravitation (12.40) evidently preserves all the properties of the simpler form (12.29), including properties 1–9 indicated earlier, with particular reference to the preservation of the global and local, interior and exterior, exact isopoincaré symmetry (Fig. 3).

The most important mathematical advancement of the former over the latter is the achievement of a geometrically and axiomatically consistent theory. In fact, isogravitation (12.29) was formulated under the specific assumption of ignoring the incomplete geometric character of Einstein's tensor, and other problematic aspects.

NOTE ADDED IN 1997

Recently Santilli (*Foundations of Physics*, **27**, 261, 1997) proved a theorem according to which *all geometries with non-null curvatures do not possess units of space and time which are invariant under the symmetries of their line elements.* As a result, curvature does not appear to allow physically un-ambiguous measurements because, e.g., one cannot conduct a meaningful measure of length with a stationary meter varying in time.

In the same memoir Santilli also proposed a geometry preserving all possible Riemannian and isoriemannian metrics, yet possessing invariant units. It is given by re-expressing the isoriemannian geometry with isometric $\hat{g} = Tg$, not with respect to the isounit $\hat{1}_{tot} = T^{-1}$, but rather with respect to the isounit $\hat{1}_{tot} = \hat{T}_{tot}{}^{-1}$ where $\hat{T}_{tot}$ is the 4×4 matrix in the factorization of of $\hat{g}$ into the Minkowskian metric η, $\hat{g} = Tg = \hat{T}_{tot}\eta$, in which case the isogeometry is isoflat, i.e., flat in isospace. He then proved that, while $\hat{1} = T^{-1}$ *is not* invariant as in ordinary geometries, the total isounit $\hat{1}_{tot} = \hat{T}_{tot}{}^{-1}$ is indeed invariant, as established by the Poincaré–Santilli isosymmetry studied later on.

As a result of the latter advances, Santilli reformulated the entire isoriemannian geometry into its isominkowskian form in which the generalized metrics are the same, but the unit is $\hat{1}_{tot}$. In this latter formulation, the entire formalism of this section remains unchanged and only the isounit is changed from $\hat{1} = T^{-1}$ to $\hat{1}_{tot} = \hat{T}_{tot}{}^{-1}$.

I.13: ISOSPECIAL RELATIVITY

We now pass to a review of *Santilli's isospecial relativities* for classical relativistic interior dynamical problems which, according to our approach (see Fig. 1 of Sect. 10), will be here presented as a particularization of Santilli's isogeneral relativities on flat isominkowski spaces, because of the prior need of Santilli's isogeometries for their true understanding.

The isospecial relativities were first proposed in Santilli (1983a) under the name of *Lorentz-isotopic relativities*, and followed the construction in the same paper of the infinite family of *Lorentz-isotopic symmetries* $\hat{O}(3.1)$ on Minkowski-isotopic spaces, now known as *Santilli's isolorentzian symmetries*. The isospecial relativities were then subjected to a second detailed analysis in Santilli (1988d), and received their final comprehensive presentation in Santilli (1991c, d).

It should be stressed that the brief review of this section is grossly insufficient to acquire a technical knowledge of these new relativities, as the reader will soon see, and the study of monographs Santilli (1991c, d) is recommended.

Remarkably, Santilli conducted alone the *construction* of the isotopic generalizations of Einstein's special relativity. In fact, despite the appearance of the nontrivial content of the paper (Santilli (1983a)), this author is aware of no additional contribution in the topic by other authors. A few independent contributions did appear in the literature, but on the *applications* of the new relativities, such as the paper by Aringazin (1989) on the "direct universality" (see Sect. 1) of the isospecial relativities; a paper by Mignani (1992) on the application to quasars of the prediction of the new relativities of the redshift of light propagating within inhomogeneous and anisotropic transparent media; and a few others.

A first independent review of Santilli's isospecial relativities and their applications has recently appeared authored by Aringazin *et al.* (1991). This review has been particularly useful for the preparation of this manuscript, and it is recommended as a useful complement of both, this book and Santilli's volumes (1991c, d).

In fact, the primary emphasis of this volume is in Santilli's novel mathematical structures, while the emphasis of Aringazin et al. (1991) is primarily on their physical applications. Also, this volume, as well as both volumes Santilli (1991c, d) are strictly classical in their content, while the review by Aringazin *et al.* (1991) presents a number of applications of Santilli's relativities in particle physics.

In this section we shall restrict our review to the classical realization of the isolorentzian symmetries, and defer their operator-matrix counterpart (Santilli (1983a), (1989c)) to some possible future volume.

Moreover, we are primarily interested in the most general possible nonlinear, nonlocal and nonhamiltonian realizations of the isolorentz symmetries on *isocurved* spaces, and then in their specialization to the *flat* isospaces.

Stated more explicitly, this section has been written along Santilli's conception as a necessary complement of the gravitational analysis of the preceding section. In fact, it provides, first, the global symmetries of exterior and interior gravitational theories and, secondly, the relativities that are applicable in the tangent planes of the interior gravitation.

In fact, we shall first review Santilli's construction of the isolorentz symmetries in their form directly applicable to gravitational theories, and then specialize them to the isotopies of the special relativity. The same approach will be followed for the subsequent review of *Santilli's isopoincaré symmetries* $\hat{P}(3.1)$.

To begin, let us recall that Santilli (1991d) classified isominkowski spaces into the following classes:

CLASS I: $M^I(x,\hat{\eta},\hat{\mathfrak{R}})$, $\hat{\eta} = T\eta$, $\eta = \mathrm{diag.}\,(1,\ 1,\ 1, -1)$, when the isospace is flat and $T > 0$.

CLASS II: $M^{II}(x,\hat{\eta},\hat{\mathfrak{R}})$, $\hat{\eta} = T\eta$, which are also flat isospaces, but the positive-definiteness of the isotopic element T is relaxed.

CLASS III: $M^{III}(x,\hat{\eta},\hat{\mathfrak{R}})$, $\hat{\eta} = T\eta$, which are curved isospaces and, as such, they *coincide* with the isoriemannian spaces of the preceding section.

DEFINITION 13.1 (Santilli (loc. cit.)): The abstract, *"Santilli's isolorentz symmetries", also called "Lorentz-Santilli symmetries", are defined on Minkowski-isotopic spaces of Class III*

$$\hat{M}^{III}{}_2(x,\hat{\eta},\hat{\mathfrak{R}}) = \hat{M}(x,\hat{g},\hat{\mathfrak{R}}):\ x = (x^\mu) = (r,\ x^4),\ \ r \in \hat{E}_2(r,\hat{G},\hat{\mathfrak{R}}),\ \ x^4 = c_o t \tag{13.1a}$$

$$x^{\hat{2}} = x^\mu \hat{g}_{\mu\nu} x^\nu = x^1 \hat{g}_{11} x^1 + x^2 \hat{g}_{22} x^2 + x^3 \hat{g}_{33} x^3 + x^4 \hat{g}_{44} x^4, \tag{13.1b}$$

$$\hat{g} = T_2\,\eta = \mathrm{Diag.}\,(\hat{g}_{11}, \hat{g}_{22}, \hat{g}_{33}, \hat{g}_{44}), \tag{13.1c}$$

$$\eta = \mathrm{diag.}\,(1,\ 1,\ 1, -1),\quad T_2 = \mathrm{diag.}\,(\hat{g}_{11}, \hat{g}_{22}, \hat{g}_{33}, -\hat{g}_{44}), \tag{13.1d}$$

$$\hat{g}_{\mu\mu} = \hat{g}_{\mu\mu}(x,\ \dot{x},\ \mu,\ \tau,\ n,\ldots) \neq 0 \ \text{ and real-valued,} \tag{13.1e}$$

$$\hat{\mathfrak{R}} = \mathfrak{R}\,\hat{1}_2,\ \hat{1}_2 = T_2{}^{-1}, \tag{5.1.f}$$

and are given by the isotransformations

$$x' = \hat{\Lambda} * x = \hat{\Lambda}\,T_2(x,\dot{x},\ldots)\,x,\ \ T_2 = \mathrm{fixed}, \tag{13.2}$$

under the conditions that they form a simple six-dimensional Lie-Santilli group $\hat{O}(3.1)$ with isotopic laws

$$\hat{\Lambda}(w)*\hat{\Lambda}(w') = \hat{\Lambda}(w')*\hat{\Lambda}(w) = \hat{\Lambda}(w+w'), \quad w \in \Re; \tag{13.3a}$$

$$\hat{\Lambda}(0) = \hat{\Lambda}(w)*\hat{\Lambda}(-w) = \hat{1}_2 = T_2^{-1}, \tag{13.3b}$$

and leave invariant isoseparation (13.1b).

The above transformations are called "abstract" because the space in which they act is not physically defined. In fact, the individual elements $\hat{g}_{\mu\mu}$ can be either positive or negative, thus characterizing either compact or noncompact groups. Santilli conceived the above definition to study the global symmetries of gravity, as well as to be effective for the classification of all possible isotopes $\hat{O}(3.1)$.

The (necessary and sufficient) conditions for isotransformations (13.2) to leave invariant isoseparation (13.1b) are given by

$$\hat{\Lambda}^t \, \hat{g} \, \hat{\Lambda} = \hat{\Lambda} \, \hat{g} \, \hat{\Lambda}^t = \hat{g}^{-1}, \tag{13.4}$$

or, equivalently,

$$\hat{\Lambda}^t \, T_2 \, \eta \, \hat{\Lambda} = \hat{\Lambda} \, T_2 \, \eta \, \hat{\Lambda}^t = \hat{1}_2 \, \eta, \tag{13.5}$$

To obtain the conditions in a more explicit form, suppose that the original Lorentz transformations $x' = \Lambda x$ are realized with the familiar expressions $x^\mu = \Lambda^\mu{}_\alpha \, x^\alpha$, e.g., as in Schweber (1962). Then, the isotopic element and isounit can be written

$$T_2 = (T_2{}^\alpha{}_\beta) = (T_{2\beta}{}^\alpha), \quad \hat{1}_2 = (\hat{1}_{2\alpha}{}^\beta) = (\hat{1}_2{}^\beta{}_\alpha), \tag{13.6a}$$

$$\hat{1}_{2\alpha}{}^\beta \, T_{2\beta}{}^\gamma = \delta_\alpha{}^\gamma. \tag{13.6b}$$

Lifting (13.2) can be written

$$x'^\mu = \hat{\Lambda}^\mu{}_\alpha \, T^\alpha{}_\beta \, x^\beta, \tag{13.7}$$

and conditions (13.4) can be written explicitly

$$\hat{\Lambda}_\alpha{}^\beta \, T_\beta{}^\rho \, \eta_{\rho\sigma} \, \hat{\Lambda}^\sigma{}_\tau = \hat{1}_\alpha{}^\delta \, \eta_{\delta\sigma}. \tag{13.8}$$

Without proof we quote the following

THEOREM 13.1 (loc. cit.): Santilli's abstract isolorentz symmetries on isospaces $\hat{M}^{III}(x,\hat{g},\hat{\Re})$ *leave invariant the isoseparation*

$$\hat{x}^{\hat{2}} = x'^{\mu}\hat{g}_{\mu\nu}x'^{\nu} = x^{\mu}\,\hat{g}_{\mu\nu}\,x^{\nu} = x^{\hat{2}}, \tag{13.9}$$

or, more explicitly,

$$x'^{\mu}\,\hat{g}_{\mu\nu}[x(x',p',..),\,p(x',p',..)]\,x'^{\nu} \equiv x^{\alpha}\,\hat{g}_{\alpha\beta}(x,p,...)\,x^{\beta}. \tag{13.10}$$

with nonsingular, Hermitean, sufficiently smooth and diagonal isometrics

$$\hat{g} = T_2\,\eta, \quad \eta \in M(x,\eta,\Re), \tag{13.11}$$

under the sole condition that the isounits $\hat{1}_2$ *are the inverse of the isotopic elements* T_2. *All the infinitely possible isosymmetries* $\hat{O}(3.1)$ *admit the connected semisimple subgroups*

$$S\hat{O}(3.1):\ \det.(\hat{\Lambda}\,\hat{g}) = +1, \tag{13.12}$$

as well as the discrete invariant subgroups

$$\hat{\Phi}(3.1):\ \mathrm{Det.}\,(\hat{\Lambda}\hat{g}) = -1, \tag{13.13}$$

and possess the following classical realization in the isocotangent bundle $T^*\hat{M}^{III}(x,\hat{g},\hat{\Re})$ *with local coordinates* $a = (a^i) = (x^{\mu}, p^{\mu})$, $\mu = 1, 2, ..., 4$, $i = 1, 2, ..., 8$ *and Lie-Santilli product*

$$[A \hat{,} B] = \frac{\partial A}{\partial x^{\mu}}\,\hat{g}^{\mu\nu}\,\frac{\partial B}{\partial p^{\nu}} - \frac{\partial B}{\partial x^{\mu}}\,\hat{g}^{\mu\nu}\,\frac{\partial A}{\partial p^{\nu}}, \tag{13.14a}$$

$$\hat{g}^{\mu\nu} = (\,|\hat{g}_{\alpha\beta}|^{-1}\,)^{\mu\nu}, \tag{13.14b}$$

1) the same (ordered set of) parameters of the conventional symmetry O(3.1), i.e., the Euler's angles θ *and Lorentz boosts* w, $u = (w_k) = (\theta, w)$, $k = 1, 2,..., 6$;

2) the same (ordered set of) generators of O(3.1)

$$J = (J_k) = (J_{\mu\nu}) = (J_{\chi}, L_{\chi}), \tag{13.15a}$$

$$J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu, \quad J_i = \epsilon_{ijl} J_{jl}, \quad L_\chi = J_{\chi 4}, \tag{13.15b}$$

$$k = 1, 2, ..., 6, \qquad \mu, \nu = 1, 2, 3, 4, \qquad \chi = 1, 2, 3,$$

3) the isocommutation rules of the Lie-isotopic algebra $\hat{O}(3.1)$ of $\hat{O}(3.1)$ in terms of brackets (13.14)

$$\hat{O}(3.1): [J_{\mu\nu} \hat{,} J_{\alpha\beta}] = \hat{g}_{\nu\alpha} J_{\beta\mu} - \hat{g}_{\mu\alpha} J_{\beta\nu} - \hat{g}_{\nu\beta} J_{\alpha\nu} + \hat{g}_{\mu\beta} J_{\alpha\nu}, \tag{13.16}$$

4) with local isocasimir invariants

$$\hat{C}^{(o)} = \hat{1}_2, \quad \hat{C}^{(1)} = (J_{\mu\nu} J^{\mu\nu}) \hat{1}_2, \quad \hat{C}^{(2)} = (\epsilon_{\mu\nu\alpha\beta} J^{\mu\nu} J^{\alpha\beta}) \hat{1}_2 \tag{5.17}$$

5) the Lie-Santilli group for the connected component

$$S\hat{O}(3.1): \quad a' = \hat{\Lambda}(u) * x =$$

$$= \{[\prod_k e_{|\xi}^{u_k \omega^{ir} \hat{1}_{2r}^{j}) (\partial_j J_k) (\partial_i)}] \hat{1}_2\} * a \stackrel{def}{=} \hat{S}_{\hat{g}}(u)\, x, \tag{13.18}$$

6) the invariant discrete subgroup $\hat{\phi}(3.1)$ characterized by the isoinversions

$$\hat{\phi}(3.1): \quad \hat{P}*x = P\,x = (-r, x^4), \tag{13.19a}$$

$$\hat{T}*x = T\,x = (r, -x^4), \quad (PT)x = (-r, -x^4), \tag{13.19b}$$

where P and T are the ordinary inversions, and
7) isosymmetries $\hat{O}(3.1)$ admit as maximal compact forms the orthogonal group in four dimension $O(4)$ as well as all its infinitely possible isotopies.

It is evident that the proof of this theorem requires a detailed knowledge of the relativistic formulation of all background mathematical tools, such as: fields, isospaces, isotransformation theory, Lie-Santilli theory, Birkhoff-Santilli mechanics, isosymmetries and the isosymplectic geometry[21] which we cannot possibly review here.

The following comments are now in order:

1) While the Minkowski space $M(x,\eta,\Re)$ with trivial unit I is unique, there exist infinitely many possible isospaces $\hat{M}^{III}(x,\hat{g},\hat{\Re})$ with isounits $\hat{1}_2 = \hat{T}_2^{-1}$ because

[21] Particularly important is the knowledge of *Santilli's relativistic isosymplectic geometry*, because important to identify the integrability conditions for brackets (13.14) to be Lie-isotopic. See Santilli (1991d).

they represent the infinitely many possible, interior, inhomogeneous and anisotropic physical media;

2) While the Lorentz symmetry O(3.1) is unique, there exist infinitely many possible isolorentzian symmetries $\hat{O}(3,1)$ characterized by the infinitely many possible isounits $\hat{1}_2$, which all possess the same dimension and simplicity of O(3.1);

3) In the same way as the Lorentz invariance cannot identify the explicit value of a Lagrangian, the invariance under Santilli's isolorentz symmetries cannot identify the isometrics, which must be computed from the given local physical conditions of the interior medium at hand;

4) While the Lorentz transformations are unique, there exist an infinite number of different isolorentzian transformations (see below for examples), characterized by the Lie-Santilli group (13.18);

5) Each of the infinitely possible isoloretzian transformations can be computed in an explicit finite form via expansion (13.18), whose convergence is assured by the assumed topological conditions (and essentially reduces to that of the conventional expansions), with the understanding that the explicit computation of the infinite series is not expected to be necessarily simple[22];

6) Each of the infinitely many isolorentzian transformations can be computed via the sole knowledge of the old parameters and generators and of the new metric (or, equivalently, of the new unit);

7) The isolorentzian transformations are formally isolinear and isolocal on $\hat{M}^{III}(x,\hat{g},\hat{\mathfrak{R}})$, but generally nonlinear and nonlocal in $M(x,\eta,\mathfrak{R})$;

8) The lifting of the conventional symmetry O(3.1) into the isotopes $\hat{O}(3.1)$ implies the generalization of the *structure constants* of the conventional formulation of Lie's theory into Santilli's *structure functions* (Sect. 6);

9) Except for the the needed topological restrictions, the isolorentzian symmetries $\hat{O}(3.1)$ leave completely unaffected the functional dependence, of the isometrics $\hat{1}_2$;

10) The classical realization of the isolorentzian symmetries can indeed admit nonlocal (integral) forms, provided that they are all embedded in the isounit $\hat{1}_2$, as permitted by the underlying isosymplectic geometry;

12) The isometrics $\hat{g}$ of isosymmetries $\hat{O}(3.1)$ can be, as particular cases, *conventional* Riemannian metrics. Therefore, Theorem 13.1 provides methods for the explicit construction of the (generally nonlinear but local) symmetries of conventional gravitational metrics such as the Schwarzschild's metric. Thus Theorem 13.1 is indeed formulated in such a way to be directly applicable to gravitation, as studied in the preceding section.

Santilli (*loc. cit.*) then studies the conditions for the local isomorphism

22 See the example of convergence to transcendental functions in the original proposal Santilli (1978a).

$\hat{O}(3.1) \sim O(3.1)$. Note that, even though isocommutation rules (13.18) appear to coincide with the conventional commutation rules of $O(3.1)$ (see, e.g., Eq. (30), p. 41 of Schweber (1962)), they are generally *different*, e.g., because the topology of the isometric $\hat{g}_{\mu\nu}$ is different than that of the Minkowski metric η.

THEOREM 13.2 (Santilli (loc. cit.)): All abstract isolorentzian symmetries $\hat{O}(3.1)$ on isospaces $M^{III}(x,\hat{g},\hat{\mathfrak{R}})$ with invariant separation (13.1b) are locally isomorphic to the conventional Lorentz symmetry $O(3.1)$ under the sole condition that the isometrics $\hat{g} = T_2\, \eta$ possess the same topological properties of the Minkowski metric η, e.g., whenever the isotopic elements T_2 or the isounits $\hat{1}_2 = T_2^{-1}$ are positive-definite; otherwise, depending on the topology of the isounits, the Lorentz-isotopic symmetries $\hat{O}(3.1)$ are locally isomorphic to any other simple six-dimensional group of Cartan's classification, such as $O(4)$ or $O(2.2)$.

Note that the positive definiteness of the isotopic element T_2 holds in a number of conventional gravitational models.

COROLLARY 13.2.1: Einstein's gravitation or any other gravitational theory (not necessarily Riemannian) with metric $\hat{g} = T\eta$, $T > 0$, admits the conventional Lorentz symmetry as a global isotopic symmetry.

Santilli then studies the explicit form of the *abstract isolorentz transformations*. The general form of the transformations for the case of the $\hat{O}(3)$ subgroups was computed in Santilli (1985b) (1988a), and it will be reviewed in Sect. 15.

We shall therefore restrict our attention only to *Santilli's abstract isolorentz boosts* in the (x^3, x^4)-plane with parameter w and generator J_{34}. Their most general possible form computed from the Lie–Santilli group (13.18) for isometrics (13.1c) is given by

$$x' = \hat{\Lambda}(w) * x = \hat{S}_{\hat{g}}(w)\, x = \tag{13.20}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(w\, \hat{g}_{33}^{\frac{1}{2}}\, \hat{g}_{44}^{\frac{1}{2}}) & -(\hat{g}_{44}/\hat{g}_{33})^{\frac{1}{2}} \sin(w\, \hat{g}_{33}^{\frac{1}{2}}\, \hat{g}_{44}^{\frac{1}{2}}) \\ 0 & 0 & (\hat{g}_{33}/\hat{g}_{44})^{\frac{1}{2}} \sin(w, \hat{g}_{33}^{\frac{1}{2}}\, \hat{g}_{44}^{\frac{1}{2}}) & \cos(w\, \hat{g}_{33}^{\frac{1}{2}}\, \hat{g}_{44}^{\frac{1}{2}}) \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{bmatrix}$$

Note that the elements of the isometric are completely unrestricted in their functional dependence in the above derivation. Note also that Santilli's abstract isolorentz transformations admit as a particular case the conventional rotations in four dimensions, trivially, for $\hat{g}_{\mu\mu} = +1$ (or $= -1$), $\mu = 1, 2, 3, 4$.

Note also that, according to the above results, *the conventional rotations in four-dimensions can also be interpreted as isolorentzian transformations in an isospace of Class III with isometric* $\hat{g} = diag\ (1, 1, 1, 1)$.

This completes our brief review of the abstract Lorentz-isotopic transformations as needed for the global symmetries of exterior and interior gravitational theories.

We now pass to the review of the particular subclass of isolorentz transformations that are physically relevant, those for the isotopies of the special relativity.

DEFINITION 13.2 (loc. cit.): Santilli's abstract $\hat{O}(3.1)$ *isosymmetries (or isotransformations) are called "general isolorentzian symmetries" when they are defined in the most general possible isospaces of Class I of the diagonal form*

$$\hat{M}^I{}_2(x,\hat{\eta},\hat{\Re}) \equiv \hat{M}^I(x,\hat{g},\hat{\Re}):\quad x^2 = x^\mu\, \hat{g}_{\mu\nu} x^\nu$$

$$= x^1 b_1{}^2 x^1 + x^2 b_2{}^2 x^2 + x^3 b_3{}^2 x^3 - x^4 b_4{}^2 x^4, \tag{13.21a}$$

$$\hat{g} = T_2 \eta,\quad \eta = \mathrm{diag.}\,(1, 1, 1, -1), \tag{13.21b}$$

$$T_2 = \mathrm{diag.}\,(b_1{}^2, b_2{}^2, b_3{}^2, b_4{}^2) > 0, \tag{13.21c}$$

$$b_\mu = b_\mu(x, \dot{x}, \mu, \tau, n, \ldots) > 0, \tag{13.21d}$$

$$1_2 = T_2{}^{-1} > 0; \tag{13.21e}$$

and they are called "restricted" when defined on the Minkowski-isotopic spaces of Class I with constant diagonal isometric

$$\hat{M}^I{}_2(x,\hat{\eta},\hat{\Re}) \equiv \hat{M}^I(x,\hat{\eta},\hat{\Re}):\quad x^2 = x^\mu\, \hat{\eta}_{\mu\nu}\, x^\nu, \tag{13.22a}$$

$$\hat{\eta} = T_2\, \eta = \mathrm{diag.}\,(b_1{}^2, b_2{}^2, b_3{}^2, -b_4{}^2), \tag{13.22b}$$

$$T_2 = \mathrm{diag.}\,(b_1{}^2, b_2{}^2, b_3{}^3, b_4{}^2) > 0, \tag{13.22c}$$

$$b_\mu = \text{constants} > 0, \tag{13.22d}$$

Similar definitions hold for the general and restricted isolorentz transformations on Minkowski-isotopic spaces of Class II.

The reader should therefore keep in mind the notation and terminology adopted by Santilli: whenever using generic elements $\hat{g}_{\mu\mu}$ we are referring to the "abstract" Lorentz-isotopic transformations; whenever using the elements $\hat{b}_\mu{}^2$ we are referring to the "general" isolorentz transformations; and, finally, whenever using the elements $b_\mu{}^2$ we are referring to the "restricted" isolorentz transformations. The most important distinction between the general and restricted isotransformations is that the former are nonlinear and nonlocal, while the latter are always linear and local. In fact, the isotransformations $x' = \Lambda * x = \hat{\Lambda} T x$ become manifestly linear and local for T = constant.

Since we now deal with a physically identified space, we can assume for parameter w its conventional physical meaning given by a speed v along the third axis, w = v. By recomputing again infinite series (13.18) for isometric (13.22b), the general isolorentz transformations on $\hat{M}^I(x,\hat{g},\Re)$ can be written (Santilli (1983a))

$$x' = \hat{\Lambda}(v) * x = \hat{S}_{\hat{g}}(v)\, x = \tag{13.23}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh(v\, \hat{b}_3 \hat{b}_4) & -(\hat{b}_4/\hat{b}_3)\sinh(v\, \hat{b}_3 \hat{b}_4) \\ 0 & 0 & -(\hat{b}_3/\hat{b}_4)\sinh(v\, \hat{b}_3 \hat{b}_4) & \cosh(v\, \hat{b}_3 \hat{b}_4) \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{bmatrix}$$

But the functional dependence of the $\hat{b}$-quantities is unrestricted in the above derivation. We have therefore the following

LEMMA 13.1 (loc. cit.): The general isolorentzian trans-formations in isospaces $\hat{M}^I(x,\hat{g},\Re)$ are given, in their broadest possible form along the third axis, by

$$x'^1 = x^1, \tag{13.24a}$$

$$x'^2 = x^2, \tag{13.24b}$$

$$x'^3 = x^3 \cosh(v\, b_3\, b_4) - x^4 \frac{b_4}{b_3} \sinh(v\, b_3\, b_4), \tag{13.24c}$$

$$x'^4 = -x^4 \frac{b_3}{b_4} \sinh(v\, b_3\, b_4) + x^4 \cosh(v\, b_3\, b_4). \tag{13.24d}$$

The proof that the isotransformations (13.24) are indeed a particular case of broader isotransformations (13.21), is an instructive exercise for the interested reader, because it implies delicate topological aspects in the transition from compact to noncompact settings.

The local isomorphism between isotransformations (13.24) and the conventional Lorentz transformations is evident, owing to the positive-definiteness of the b-quantities. In fact, we have the following property.

THEOREM 13.3 (loc. cit.): All infinitely possible, general, isolorentzian symmetries $\hat{O}(3.1)$ on isominkowski spaces of Class I, Eq.s (13.2), are locally isomorphic to the conventional Lorentz symmetry O(3.1).

We now pass to the review of Santilli's restricted isolorentzian transformations (Definition 13.2). It is easy to see that when the b-quantities of isotransformations (13.24) are constants, we have the following

COROLLARY 13.3.1 (loc. cit.) : All infinitely possible restricted isolorentzian transformations in the (3-4)-plane of Minkowski-isotopic spaces of Class I with constant diagonal isometrics can be written in the form [23]

$$x'^1 = x^1, \tag{13.25a}$$

$$x'^2 = x^2, \tag{13.25b}$$

$$x'^3 = \hat{\gamma}\,(x^3 - \beta x^4), \tag{13.25c}$$

$$x'^4 = \hat{\gamma}\,(x^4 - \hat{\beta} x^3), \tag{13.25d}$$

[23] It is an instructive exercise for the reader interested in learning Santilli's isospecial relativity that the use of the different quantities β for Eq. (13.25c) and $\hat{\beta}$ for Eq. (13.25d) is correct and not a misprint.

where

$$\beta = v/c_o, \quad \hat{\beta} = \frac{vb_3}{c_o b_4}, \qquad \hat{\beta}^2 = \frac{v^k b_k{}^2 v^k}{c_o b_4{}^2 c_o}, \tag{13.26a}$$

$$\cosh(v\, b_3\, b_4) = \hat{\gamma} = (1 - \hat{\beta}^2)^{-\frac{1}{2}}, \tag{13.26b}$$

$$\sinh(v\, b_3\, b_4) = \hat{\beta}\, \hat{\gamma}. \tag{13.26c}$$

The following property should be recalled here.

COROLLARY 13.2.2 (loc. cit.): The general (or restricted) isolorentzian symmetries $\hat{O}(3.1)$ on isospaces $\hat{M}^I(x,\hat{g},\hat{\mathfrak{R}})$ can reconstruct as exact at the isotopic level all conventional breaking of the Lorentz symmetry, under the sole condition that the underlying generalized metrics $\hat{g} = T_2\,\eta$ preserve the topology of the conventional Minkowski metric η, i.e., $T_2 > 0$.

An example of the restricted isolorentz metric is provided by the deformation of the Minkowski metric worked out by H. B. Nielsen and I. Picek (1983) in the interior of pions and kaons, which resulted in the following deformation (*"mutation"* in Santilli's terminology) of the interior Minkowski metric

$$T_2 = \mathrm{diag.}\,[(1 - \alpha/3), (1 - \alpha/3), (1 - \alpha/3), (1 + \alpha)] \tag{13.27}$$

where the α-quantity was called by the Nielsen and Picek the *"Lorentz-asymmetry parameter"*. As one can see, the Lorentz symmetry is exact for metric (13.27), provided that it is not realized in terms of the simplest conceivable Lie product, but in terms of Santilli's lesser trivial isotopic product (13.14). Similar results hold for all possible physically achievable *mutations* of the Minkowski metric, those of Class I. Theorem 13.3 can therefore be called a *technique for reconstructing the exact Lorentz symmetry when believed to be conventionally broken.*

It should be indicated that, while the Lorentz symmetry remains exact for all interior generalizations of the Minkowski metric of Class I, this is evidently not the case for isospaces of Class II and III.

It is an instructive exercise for the reader interested in learning Santilli's relativities to compute other isolorentzian transformations for given explicit functional dependence of the characteristic b-functions (see Sect. 15 for the

isorotational subgroup).

It is also instructive to show that the abstract, general and special isotransformations do indeed leave invariant isoseparation (13.1b), (13.21a) and (5.22a), respectively. It is finally suggested to the interested reader to verify that *all general isolorentz transformations in the (3,4) plane can be cast in form (6.25).* We can equivalently say that *the geometrization of the interior dynamical problem characterized by $\hat{O}(3.1)$ on isospaces $\hat{M}^I(x,\hat{g},\hat{\Re})$ can be unified in form (13.25).*

The following important property is a consequence of the "direct universality" of Birkhoffian mechanics (Sect. 6), as well as of the arbitrariness of the $\hat{b}$-functions.

COROLLARY 13.2.3 (loc. cit.): The general Lorentz-isotopic symmetries, and related isotransformations, are "directly universal", in the sense that they admit as particular cases all possible generalizations of the Lorentz symmetry and related transformations characterized by topology-preserving, linear or nonlinear, and local or nonlocal deformations of the Minkowski metric ("universality"), directly in the frame of the observer ("direct universality").

Santilli (1991d) then shows that all available preceding attempt at generalizing Einstein's special relativity are particular cases of his isospecial relativities.

The most notable example is the apparently first, true generalization of the special relativity achieved by Bogoslovwski (1977), (1984), which is called by Santilli (loc. cit.) *Bogosklovwski's special relativity*, and the same terminology is adopted in this book.

In essence, Bogoslowski generalized the special relativity for homogeneous but anisotropic conditions, although refereed to space iself. Also, his generalization is based on conventional Lie techniques.

Santilli's covering isospecial relativities bring Bogoslowski's special relativity into a new light, and point out some of its possibilities that were grossly ignored by the physics community[24]. In fact, they show that the anisotropy of the latter can indeed be referred to interior physical media with very intriguing applications, such as the possible confinement of quarks (see, e.g., Preparata (1981)). Moreover, the Lie-Santilli's theory permits a vast simplification of the rather complex derivation by Bogoslovski of his anisotropic generalization of the Lorentz transformations.

The fact that Bogoslowski's homogeneous but anisotropic formulations are a particular case of Santilli's inhomogeneous and anisotropic theories, is evident.

We finally recall that the Lorentz-isotopic symmetries of this section are a

[24] Despite the manifest value of Bogoslowski's studies, this author is aware of no significant, independent analysis of his relativity that has appeared in the literature, besides those by Santilli (1988c), (1991d) and very few others.

particular case of Santilli's (1981a) broader *Lorentz-admissible symmetries* . The former symmetries are recommendable for the relativistic characterization of closed-isolated composite systems such as Jupiter (Fig. 1 of Sect. 10) with *nonlinear*, nonlocal *and* nonhamiltonian interior forces. The latter symmetries are more effective for the description of a relativistic, extended test particle moving within a physical medium, resulting in nonconservative conditions due to the most general known external forces (Appendix E).

We now review the classical realization of *Santilli's (1988c), (1991d) inhomogeneous Lorentz-isotopic symmetries*, also called *Poincare'-isotopic symmetries* , or *isopoincaré symmetries* $\hat{P}(3.1)$.

Consider a set of N particles denotes with the symbol a = 1, 2,...., N, in isominkowski spaces $M^{III}{}_2(x,\hat{\eta},\mathfrak{R}) = \hat{M}^{III}(x,\hat{g},\hat{\mathfrak{R}})$ with local separation

$$x_{ab}{}^{\hat{2}} = (x_a{}^{\mu} - x_b{}^{\mu})\,\hat{g}_{\mu\nu}(x, v, a, \mu, \tau, n, ...)\,(x_a{}^{\nu} - x_b{}^{\mu}). \tag{13.28}$$

DEFINITION 13.3 (loc. cit.): Santilli's "abstract isopoincaré symmetries" $\hat{P}(3.1)$ *are the largest possible, ten-dimensional isotopic group of isometries of isoseparation (13.28) which are isolinear and isolocal on isominkowski spaces* $M^{III}(x,\hat{g},\hat{\mathfrak{R}})$, *but nonlinear and nonlocal when projected on the conventional Minkowski space* $M(x,\eta,\mathfrak{R})$. *The "general isopoincaré symmetries" are the most general possible, isolinear and isolocal isosymmetries of isoseparation (13.28) on isominkowski spaces* $\hat{M}^{I}(x,\hat{g},\hat{\mathfrak{R}})$. *Finally, the "restricted isopoincaré symmetries" are the most general possible, linear and local isometries of isoseparation (13.28) on isominkowski spaces* $\hat{M}^{I}(x,\hat{\eta},\hat{\mathfrak{R}})$, *with isometric* $\hat{\eta}$ *independent from the local coordinates and all their derivatives.*

As is well known (see, e.g., Schweber (1962)), the conventional Poincaré group possesses the structure of the semidirect product

$$P(3.1) = O(3.1) \otimes T(3.1), \tag{13.29}$$

where T(3.1) is the (Abelian) invariant subgroup of translations in Minkowski space.

The conventional Poincaré transformations are given by the well known *linear and local transformations* on $M(x,\eta,\mathfrak{R})$

$$x' = \Lambda x + x^{\circ}, \qquad \Lambda \in O(3.1)\,, \; x^{\circ} = (x^{\circ\mu}) \in \mathfrak{R}, \tag{13.30}$$

A classical realization of the Poincare' simmetry for the case of N particles with non-null masses is given by the ten (ordered) parameters

$$w = (w) = (\theta, u, x^{\circ}), \quad k = 1, 2, \dots, 10, \tag{13.31}$$

and generators in $T^*M(x,\eta,\Re)$

$$X = (X_k) = (J_{\mu\nu}, P_\mu), \; k = 1, 2, \dots, 10, \tag{13.32a}$$

$$J_{\mu\nu} = \sum_a x_{a\mu} p_{a\nu} - x_{a\nu} p_{a\mu}, \quad P_\mu = \sum_a p_{a\mu}, \tag{13.32b}$$

with Lie algebra P(3.1) characterized by the commutation rules in terms of brackets (IV.4.16)

$$P(3.1): [J_{\mu\nu}, J_{\alpha\beta}] = \eta_{\nu\alpha} J_{\beta\mu} - \eta_{\mu\alpha} J_{\beta\nu} - \eta_{\nu\beta} J_{\alpha\mu} + \eta_{\nu\beta} J_{\alpha\nu}, \tag{13.33a}$$

$$[J_{\mu\nu}, P_\alpha] = \eta_{\mu\alpha} P_\nu - \eta_{\nu\alpha} P_\mu, \tag{13.33b}$$

$$[P_\mu, P_\nu] \quad 0, \qquad \mu, \nu, \alpha = 1, 2, 3, 4, \tag{13.33c}$$

with Casimir invariants

$$C^{(o)} = 1, \quad C^{(1)} = P^2 = P^\mu P_\mu = P^\mu \eta_{\mu\nu} P^\nu \tag{13.34a}$$

$$C^{(2)} = W^\mu W_\mu = W^\mu \eta_{\mu\nu} W^\nu, \quad W_\mu = \epsilon_{\mu\alpha\beta\rho} J^{\alpha\beta} P^\rho, \tag{13.34b}$$

Lie group structure

$$P(3.1): \; a' = \{\textstyle\prod_k e|_\xi^{\, w_k \, \omega^{ij} (\partial_j X_k)(\partial_i)}\} \, a, \tag{13.35}$$

and discrete invariant subgoup

$$\phi(3.1): \; Px = (-r, x^4), \; Tx = (r, -x^4), \; PTx = (-r, -x^4), \tag{13.36}$$

The following isotopic liftings then occur.

THEOREM 13.4 (loc. cit.): The Poincare' symmetry P(3.1) *on the conventional Minkowski space* $M(x,\eta,\Re)$ *admits an infinite number of abstract isopoincaré generalizations* $\hat{P}(3.1)$ *on the infinite number of corresponding isominkowski spaces* $\hat{M}^{III}(x,\hat{g},\hat{\Re})$, *with diagonal, nonsingular and Hermitean isometrics* $\hat{g} = T_2\,\eta$,

and isounit $\hat{1}_2 = T_2^{-1}$. *All isosymmetries* $\hat{P}(3.1)$ *admit the decomposition into the semidirect product*

$$\hat{P}(3.1) = \hat{O}(3.1) \otimes \hat{T}(3.1), \tag{13.37}$$

where the subgroups $\hat{O}(3.1)$ *are the (simple) isolorentzian subgroups of Theorem 13.1,*

$$\hat{O}(3.1): \quad x' = \hat{\Lambda} * x = \hat{\Lambda} T_2 x, \quad \hat{\Lambda}^t \hat{g} \hat{\Lambda} = \Lambda \hat{g} \hat{\Lambda}^t = \hat{g}^{-1}, \tag{13.38}$$

and the groups $\hat{T}(3.1)$ *are the (isoabelian) invariant subgroups of isotranslations*

$$\hat{T}_a(3.1): \quad x'^{\mu} = \hat{T}(x^{\circ}) * x^{\mu} = x^{\mu} + x^{\circ\mu}\, b^{\mu\,-2}(x, \dot{x}, \mu, \tau, n, ...), \tag{13.39a}$$

$$p'^{\mu} = \hat{T}(x^{\circ}) * p^{\mu} = p_{\mu}, \tag{13.39b}$$

where the b*-functions are generally nonlinear and nonlocal in all their arguments to be identified below.*
All isotopes $\hat{P}(3.1)$ *admit the following classical realization for a system of N particles in the isocotangent bundle* $T^*\hat{M}^{III}(x,\hat{g},\hat{\Re})$ *with local charts* $a = (a^i) = (x, p) = (x^{a\mu}, p^{a\mu})$, $i = 1, 2, ..., N$:

1) The same ordered set of generators (13.31) of the conventional symmetry;
2) The same (ordered set of) generators (13.32) of the conventional symmetry;
3) the isocommutation rules in terms of the isotopic brackets (13.14)

$$\hat{P}(3.1): [J_{\mu\nu} \hat{,} J_{\alpha\beta}] = \hat{g}_{\nu\alpha} J_{\beta\mu} - \hat{g}_{\mu\alpha} J_{\beta\nu} - J_{\beta} J_{\alpha\mu} + \hat{g}_{\mu\beta} J_{\alpha\nu}, \tag{13.40a}$$

$$[J_{\mu\nu} \hat{,} P_{\alpha}] = \hat{g}_{\mu\alpha} P_{\nu} - \hat{g}_{\nu\alpha} P_{\mu}, \tag{13.40b}$$

$$[P_{\mu} \hat{,} P_{\nu}] = 0, \quad \mu, \nu = 1, 2, 3, 4 \tag{13.40c}$$

4) the local[25] *isocasimir invariants*

[25] This local restriction is due to insufficient knowledge of the isoneutral elements (end of Sect. 6). Note that the isocasimirs are elements of the field, as they should be. However, the reader not familiar with isotopic techniques should keep in mind that, for the case of *matrix* representations, the entire isocasimir is an invariant including the isounit (see, e.g., Santilli (1983a)). However, for the case of classical realizations via *functions* in isospaces, only the functions multiplying the isounits are isoinvariants. In fact, even in the classical case, the isounits are matrices.

$$\hat{C}^{(o)} = \hat{1}_2 = \hat{T}_2^{-1}, \tag{13.41a}$$

$$\hat{C}^{(1)} = \hat{P}^2 = (P^{\mu}\, \hat{g}_{\mu\nu}\, P^{\nu})\,\hat{1}_2 = (P^{\mu}\, P_{\mu})\,\hat{1}_2, \tag{13.41b}$$

$$\hat{C}^{(2)} = \hat{W}^2 = (W^{\mu}\, \hat{g}_{\mu\nu}\, W^{\mu})\,\hat{1}_2, \quad W_{\mu} = \epsilon_{\mu\alpha\beta\rho}\, J^{\alpha\beta}\, P^{\rho}, \tag{13.41c}$$

5) The Lie–Santilli group structure for the connected part $\hat{P}_{\circ}(3.1) = S\hat{O}[3.1) \otimes \hat{T}(3.1)$ *on* $T^*\hat{M}^{III}(x,\hat{g},\hat{\Re})$

$$a' = \{\hat{\Lambda}, \hat{T}\} * a = \{[\, e_{\xi}^{\, w_k\, \omega^{ir}\, I_{2r}{}^{j}\, (\partial_j X_k)\, (\partial_i)}\,]\, \hat{1}_2\} * a =$$

$$= \{S_{\hat{g}}(\theta,u),\ T_{\hat{g}}(x^{\circ})\}\, a, \tag{13.42}$$

6) the $\tilde{b}$*–functions are explicitly given by*

$$\tilde{b}_{\mu}^{\ -2} = \hat{b}_{\mu}^{\ -2} + x^{\alpha}\,[\hat{b}_{\mu}^{\ -2} \hat{,}\ P_{\alpha}]/2! + x^{\circ\,\alpha}\, x^{\circ\,\beta}\,[[\hat{b}_{\mu}^{\ -2} \hat{,}\, P_{\alpha}] \hat{,}\ P_{\beta}]/3! + \ldots\ldots \tag{13.43}$$

and

7) the invariant isodiscrete component is the same as that in $\hat{O}(3.1)$, *i.e.,*

$$\hat{\phi}(3.1):\ \hat{P}*x = (-r, x^4),\quad \hat{T}*x = (r, -x^4),\quad (\hat{P}*\hat{T})*x = (-r, -x^4).$$

(

Despite the manifest similarities between the conventional Poincare' symmetries and all its isotopes, the latter have non-trivial physical implications, as illustrated by the loss of the conventional Poincaré transformations, with consequential need to generalize Einstein's special relativity (see below).

The following property is by now evident.

COROLLARY 13.4.1 (loc. cit.): All infinitely possible general and restricted isopoincaré symmetries $\hat{P}(3.1)$ *on isominkowski spaces* $M^I(x,\hat{g},\hat{\Re})$, *are locally isomorphic to the conventional Poincaré symmetry* $P(3.1)$.

It is understood that on isospaces of Class II or III the above isomorphism are no longer guaranteed. As an example, the use of the classification of $\hat{O}(3.1)$ (Sect. 6) shows that some of isotopes $\hat{P}(3.1)$ on isospaces of Class II are locally isomorphic to $O(4)\times T(4)$ or $O(2.2)\times T(2.2)$. This illustrates the need that, for all physical applications, the isotopic elements T are positive-definite, thus resulting either in the isoriemannian spaces $M^{III}(x,\hat{g}(\hat{\Re})$ or in the flat isospaces $M^I(x,\hat{\eta},\hat{\Re})$ of the

isospecial relativity (see below).

It is evident that the general and restricted isolorentz and isopoincaré symmetries constitute isotopic coverings of the corresponding conventional symmetries in the sense indicated earlier.

Note the explicit dependence of the isometric on the local coordinates x. Also, the composition law of two isopoincaré transformations $\{\hat{\Lambda}_k , \hat{T}_k\}$, k = 1, 2, is given by

$$\{\hat{\Lambda}_1 , \hat{T}_1\} * \{\hat{\Lambda}_2 , \hat{T}_2\} = \{\hat{\Lambda}_1 * \hat{\Lambda}_2 , \hat{T}_1 + \hat{\Lambda}_2 * \hat{T}_2\}. \tag{13.45}$$

Moreover, one should note for future use the isotopy of the group of translations

$$T(3.1) = e_{|\xi}^{x^{\circ\mu} \eta_{\mu\nu} p^\nu} \Rightarrow \hat{T}(3.1) = e_{|\xi}^{x^{\circ\mu} \hat{g}_{\mu\nu} p^\nu}, \tag{13.46}$$

which is at the foundation of Santilli (1983a) notion of *electromagnetic wave propagating within an inhomogeneous and anisotropic physical medium*.

As it is the case for the conventional and isotopic Lorentz symmetries, the isopoincaré symmetries are not freely defined in isospaces $\hat{M}^{III}(x,\hat{g},\hat{\mathfrak{R}})$, but rather on the hypersurface of the constraints (see Santilli (1991d)).

Finally, the Poincaré–isotopic symmetries of this section are a particular case of *Santilli's (1981a) Poincaré–admissible symmetries*, the latter being the most general possible symmetries for extended–deformable particles under the most general known nonconservative dynamical conditions.

Santilli (1991d) then illustrates the physical implications of the isotopic liftings of the Lorentz and Poincaré symmetries by identifying their implications for the characterization of *particles* . In turn, this provides a number of experimental grounds, first, for the identification of the physical conditions under which the isotopies are applicable and, second, for the verification of the quantitative predictions of the novel theory in the arena of its applicability.

As well known, Einstein's special relativity characterizes particles as *massive points*. But point are perennial geometrical objects. Thus, *according to contemporary relativistic views, elementary particles preserve their intrinsic characteristics for all conceivable physical conditions in the Universe.*

In his limpid writings, Einstein (1905) avoided such a manifestly excessive assumption, because he identified quite clearly the arena of applicability of his theories. Santilli therefore assumes as exact Einstein's views, but not necessarily those of his contemporary followers. In particular, *throughout his analysis Santilli assumes that elementary particles preserve their intrinsic characteristics under Einsteinian conditions*, i.e.,

1) particles can be well approximated as being massive points;

2) when moving in the homogeneous and isotropic vacuum (empty space);

3) while experiencing only action-at-a-distance, local-potential (selfadjoint) forces.

In this volume we are interested in studying particles in *Santilli's conditions*, i.e.,

1) particles (and/or their wavepackets) which cannot be approximated as being point-like, but require a representation of their actual extended, and therefore deformable shape;

2) when moving within generally inhomogeneous and anisotropic physical media;

3) while experiencing conventional, action-at-a-distance, local-potential forces, as well as contact, short range nonlinear, nonlocal and nonhamiltonian interactions with the medium itself.

A quantitative way of expressing the above notion is given by the following

DEFINITION 13.4 (Santilli (1988c, (1991d)): A classical relativistic isoparticle is a representation of one of the infinitely possible isopoincaré symmetries in Minkowski-isotopic spaces $\hat{M}^I(x,\hat{g},\hat{\mathfrak{R}})$,

$$\hat{P}(3.1):\ x' = \{\hat{\Lambda}(\theta,u)\ ,\ \hat{T}(x^\circ)\} * x = (\hat{\Lambda}(\theta,u)\ ,\ \hat{T}(x^\circ)\} T_2\, x$$

$$= \{\prod_k e|_\xi^{\ w_k\, \omega^{ir}\, \hat{1}_{2r}{}^{j}\, (\partial_\nu X_k)\, (\partial_i)}\, \hat{1}_2\} * x, \tag{13.47a}$$

$$w = (u,x^\circ), \quad X = (J_{\mu\nu}\, , P_\mu), \tag{13.47b}$$

$$\hat{M}^I(x,\hat{g},\hat{\mathfrak{R}}):\quad \hat{g} = T_2\, \eta, \quad \hat{\mathfrak{R}} = \mathfrak{R}\hat{1},\ \hat{1}_2 = \hat{T}_2^{\ -1} > 0, \tag{13.47c}$$

Equivalently, Santilli's classical nonrelativistic isoparticles are the generalization of the corresponding classical Einsteinian particles characterized by the isotopic lifting of the trivial unit I of the conventional Poincaré symmetry, into one of the infinitely possible isounits $\hat{1}_2 > 0$ *of the isopoincaré symmetries* $\hat{P}(3.1)$.

The primary and most visible consequences of Santilli's generalized notion of particle is that <u>*an isoparticle is no longer unique, but can exist with an infinite number of different intrinsic characteristics*</u>[26].

[26] The reader should not strictly refer the consideration at this point to an elementary particle in quantum mechanics, because the formulations here are purely classical (although all the classical features are magnified under quantization, see Santilli (1989, a, b, c, d)).

The best way to illustrate the above possibilities, is by considering an actual, physical, extended test particle, such as a charged perfect sphere of a given radius moving in vacuum under an external electromagnetic field. Under these conditions, the test particle can be well approximated as being a massive point, and fully treated via the Poincaré symmetry, trivially, because the actual shape of the particle does not affect its dynamical evolution.

Suppose now that the above test particle penetrates within a physical medium, such as an atmosphere, at relativistic speeds. Then, the particle experiences a deformation of its shape, say, into a prolate ellipsoid. Moreover, one can see that the deformed shape is not unique, but depending on the local physical conditions, such as pressure, density, speed, etc.

Thus, while the test particle has a unique configuration in vacuum (Einstein's conditions), the same test particle has an infinite number of different configurations when moving within a physical medium (Santilli's conditions).

Santilli (1978b) conceived his isotopies of the Lorentz and Poincaré symmetries precisely to be able to represent the actual shape of the particle considered, as well as all its infinitely possible deformations.

To make the reader aware of the nontriviality of the discoveries herein reviewed, let us recall that the total magnetic moments of nuclei are still basically unexplained at this writing, despite over fifty years of research. This is likely due to the point-like abstraction of protons and neutrons which is inherent in the very structure of contemporary theories.

Santilli (1989a, b, c, d), (1991d) has shown that, once the proton and neutrons are represented as they are in physical reality, extended charge distributions of about 1F ($= 10^{-13}$ cm), their deformability under sufficiently intense external fields and/or collisions is consequential. In turn, the deformation of the charge distribution of neutrons and protons implies a necessary consequential mutation of their intrinsic magnetic moments.

The applicability of Santilli's concept of isoparticle offers the possibility of resolving the problem of the total nuclear magnetic moments via their mere reduction to a (very small() deformations of the charge distribution of the protons and neutrons, with consequential (generally small) alteration of their intrinsic magnetic moments under nuclear conditions.[27]

For this and numerous other applications of Santilli's isoparticle, we are regrettably forced to refer the interested reader to the locally quoted literature. We

[27] This point is recalled here because, as stressed by Santilli (1989a, b, c), it is an essentially *classical* issues. We are referring here to the prediction of Maxwell electrodynamics of the alteration of the magnetic moment of an extended, charged and spinning sphere when its shape is deformed by external forces. In this way, the problem of total nuclear magnetic moments can be first approach on purely classical grounds. Operator formulations should only be expected to provide refinements of the classical results.

also refer the reader to the excellent review by Aringazin *et al.* (1991) of Rauch's experiments via neutron interferometry providing preliminary experimental information supporting the possible alterations of the charge distribution of neutrons with the consequential alteration of their magnetic moments.

The necessary conditions for the existence of Santilli's mutations were identified in the original proposal (1978b) and can be now formulated as follows.

PROPOSITION 13.1 (Santilli (1988c)): A necessary condition for the mutation of the intrinsic characteristics of particles is that the local physical conditions at hand imply a violation of the conventional Lorentz symmetry of nonlinear, or nonlocal or nonhamiltonian type.

Santilli (1991d) then identifies the necessary and sufficient conditions for a mutation. An inspection of the invariants of the isotopies $P(3.1) \Rightarrow \hat{P}(3.1)$,

$$P^2 = P^\mu \eta_{\mu\nu} P^\nu \Rightarrow \hat{P^2} = P^\mu \hat{g}_{\mu\nu} P^\nu, \tag{13.48a}$$

$$W^2 = W^\mu \eta_{\mu\nu} W^\nu \Rightarrow \hat{W^2} = W^\mu \hat{g}_{\mu\nu} W^\nu, \quad W_\mu = \epsilon_{\mu\alpha\beta\gamma} J^{\alpha\beta} P^\gamma \quad , \tag{13.48b}$$

yields the following:

PROPOSITION 13.2 (loc. cit.): A necessary and sufficient condition for the mutation of the intrinsic characteristics of elementary particles is that the isometric $\hat{g}$ is a nontrivial isotopy of the Minkowski metric η, $\hat{g} = T_2\, \eta$, $T_2 \neq I$, $T_2 > 0$.

Finally, the reader should not assume that, under mutation, we lose fundamental space-time symmetries. In fact, *Santilli defines isoparticles via the exact Poincaré symmetry.* The only difference is that, while Einstein's special relativity implies the simplest conceivable realization of the Lie products, Santilli assumes a lesser trivial realization. This leads to the following restriction on the mutations evidently imposed by the local isomorphism $P(3.1) \approx \hat{P}(3.1)$.

PROPOSITION 13.3 (Santilli (loc. cit.)): All infinitely possible configurations of Santilli' isoparticles coincide with the conventional Einsteinian form of the same particles at the abstract, realization-free level.

In different terms, mutations are not arbitrary, but geometrically restricted to such a class that conventional and mutated intrinsic characteristics can be unified at the abstract, realization-free level.

Further advances in the topic require the isorepresentations of $\hat{P}(3.1)$ which have not been studied at this writing.

The study of isoparticles also requires the relativistic generalization of the various methods studied in Sections 1 to 9, which we cannot possibly review here.

We merely mention the case of a "free isoparticle", which is characterized by the isometric of the constant form

$$\hat{\eta} = \hat{g} = \text{diag.}\,(b_1{}^2, b_2{}^2, b_3{}^2, -b_4{}^2), \qquad b_\mu = \text{constants} > 0, \tag{13.49}$$

and the Hamiltonian

$$H = p^\mu\,\hat{\eta}_{\mu\nu}\,p^\nu/2\lambda \;-\; \tfrac{1}{4}c^4\lambda \tag{13.50}$$

(where λ is a Lagrange multiplier for the invariance constraint), resulting in the equations of motion

$$\dot{x}_\mu = b_\mu{}^{-2}\,\partial H/\partial p^\mu = p_\mu/\hat{m}, \quad \text{(no sum)} \tag{13.51a}$$

$$\dot{p}_\mu = -\,b_\mu{}^{-2}\,\partial H/\partial x^\mu = 0, \quad \text{(no sum)} \tag{13.51b}$$

$$\lambda = \dot{x}^\mu\,\hat{\eta}_{\mu\nu}\,\dot{x}^\nu = -1. \tag{13.51c}$$

which are manifestly $\hat{P}(3.1)$–isocovariant, nevertheless, they coincide with the conventional equations for a free particle.

One can then see the capability of Santilli's isopoincaré symmetries to represent at this purely classical level the actual shape of the particle via the b–quantities, evidently in space–time, as well as as to represent all its infinitely possible mutations, e.g., via a suitable functional dependence on the b–quantities. By comparison, Einstein's special relativity can achieve an indirect representation of the shape of the particle only after the second quantization, the actual shape cannot be represented, and all possible deformations are evidently eluded in order not to violate the pillar of quantum mechanics, the O(3) symmetry.

A virtually endless number of local-potential, as well as nonlocal nonpotential generalizations of the above case exist. We only recall the case of a charged isoparticle under an external (evidently conventional) electromagnetic field, which is representable by the following *relativistic Hamilton–Santilli equations*

$$\dot{x}_\mu = b'_\mu{}^{-2}\,\frac{\partial H}{\partial p^\mu} = (p_\mu - \frac{e}{c}A_\mu)\,/\,m, \tag{13.52a}$$

$$\dot{p}_\mu = - b'_\mu{}^{-2} \frac{\partial H}{\partial x^\mu} = \frac{e}{cm}(\partial_\mu A_\alpha)(p^\alpha - \frac{e}{c} A^\alpha), \tag{13.52b}$$

$$\phi = \frac{\partial H}{\partial \lambda} = - \frac{1}{2\lambda^2}(p - eA/c)^2 - \tfrac{1}{2} c^4 = 0, \ \lambda = m, \tag{13.52c}$$

$$H = (p^\mu - eA^\mu / c) \hat{\eta}_{\mu\nu} (p^\nu - eA^\nu)/2\lambda - \tfrac{1}{2}c^4\lambda, \tag{13.52d}$$

It is easy to see that the above system is invariant with respect to the isopoincaré symmetry, but it is manifestly noninvariant under the conventional Poincaré symmetry. The system can be easily generalized to represent an extended and deformable charge under the most general known interactions, those of linear and nonlinear, local and nonlocal, and selfadjoint as well as nonselfadjoint type.

We refer the interested reader to Santilli (1991d) for brevity.

We now briefly outline *Santilli's isospecial relativities* , which were originally submitted in Santilli (1983a) under the name of *Lorentz-isotopic relativities*, and then developed in Santilli (1988c), (1991d).

Let us begin with the following:

DEFINITION 13.5 (loc. cit.): Santilli's "general isospecial relativities" are given by the generalizations of Einstein's special relativity characterized by the general isolorentz and isopoincare' symmetries of Theorems 13.1 and 13.4, namely, by their most general possible, nonlinear, nonlocal and nonhamiltonian realizations on Minkowski-isotopic spaces $\hat{M}^I(x,\hat{g},\hat{\mathfrak{R}})$, *or, equivalently, by the most general possible, nonlinear and nonlocal realizations of the isounits* $\hat{1}_2 > 0$. *Santilli's "restricted isospecial relativities" are characterized instead by the most general possible linear and local isolorentz and isopoincaré transformations on isospaces* $\hat{M}^I(x,g,\hat{\mathfrak{R}})$ *with isometric independent from the local variables and their derivatives.*

The reader should be aware from the beginning that, by central conception, *Santilli's and Einstein's relativities coincide at the abstract level by construction* (see later on Theorem 13.5). In particular, as we shall see momentarily, the main postulates of Santilli's relativities are given by an isotopy of the corresponding postulates of the special relativity.

Thus, *all the deviations from conventional settings predicted by Santilli's relativities are, in the final analysis, a direct manifestation of the abstract axioms of Einstein's special relativity*.

To put it explicitly, *in case any of the prediction of Santilli's isospecial relativities is disproved by future experiments, this will likely imply* a *revision of the axiomatic structure of Einstein's special relativity.*

With a clear understanding on these premises, let us now review the five basic postulates of Santilli's isospecial relativities on the following isospaces

$$\hat{M}^I(x,\hat{g},\hat{\Re}): \quad x^{\hat{2}} = x^\mu \hat{g}_{\mu\nu} x^\nu)$$

$$= x^1 \hat{b}_1{}^2 x^1 + x^2 \hat{b}_2{}^2 x^2 + x^3 \hat{b}_3{}^2 x^3 - x^4 \hat{b}_4{}^2 x^4,$$

$$= r^k \hat{b}_k{}^2 r^k - t\, c_o \hat{b}_4{}^2 t = r^{\hat{2}} - t\, c^2\, t, \tag{13.53a}$$

$$x = (r, x^4) = (r, c_o t), \quad r \in \hat{E}_2(r,\hat{G},\hat{\Re}) \tag{13.53b}$$

$$\hat{g} = T_2\, \eta, \tag{13.53c}$$

$$\eta = \mathrm{diag.}\,(1,1,1,-1) \in M(x,\eta,\Re), \tag{13.53d}$$

$$T_2 = \mathrm{diag.}\,(\hat{b}_1{}^2, \hat{b}_2{}^2, \hat{b}_3{}^2, \hat{b}_4{}^2) > 0, \quad \hat{\Re} = \Re\, \hat{1}_2, \quad \hat{1}_2 = T_2{}^{-1}, \tag{13.53e}$$

$$\hat{b}_\mu = \hat{b}_\mu(x, u, a, \mu, \tau, n, \ldots) > 0, \quad \mu = 1, 2, 3, 4, \tag{13.53f}$$

$$\hat{b}_1 = \hat{b}_2 = \hat{b}_3 = \hat{b}, \tag{13.53g}$$

$$c = c_o \hat{b}_4, \tag{13.53h}$$

where: conditions (13.53g) are assumed for the specific purpose of identifying the relativistic effects of the interior dynamical problem, and separate them from the effects due to isorotations $\hat{O}(3)$ reviewed in Sect. 15; conditions (13.53f) are assumed, in addition to the conditions $\hat{b}_\mu{}^2 > 0$, to permit the identification of the b-functions with physical quantities; the quantities $\hat{b}_\mu$ have the most general possible nonlinear and nonlocal dependence in all permitted variables and quantities; the metric $\hat{g}$ is that of the isocotangent bundle $T^*\hat{M}^I(\hat{x},\hat{\eta},\hat{\Re})$, i.e., it is such that brackets (13.14) characterized by the inverse $\hat{g}^{-1}$ verify the classical Lie-Santilli axioms (Sect. 6); and Eq. (13.53h) represents the geometrization of the conventional speed of light in vacuum c_o, as characterized by Santilli's isotopies.

The analysis for the restricted isotopies will be conducted in the isospaces of the particular form

$$\hat{M}^I(x,\hat{\eta},\hat{\Re}) = \hat{M}^I{}_o(x,\hat{\eta},\hat{\Re}), \tag{13.54a}$$

$$\hat{\eta} = \text{diag.}\,(b_1{}^2, b_2{}^2, b_3{}^2, -b_4{}^2) = \text{local constant} > 0, \tag{13.54b}$$

$$b_\mu = \hat{b}_\mu = \text{constants} > 0, \quad b_1 = b_2 = b_3 = b, \quad c = c_o b_4. \tag{13.54c}$$

Let us begin by studying the invariant speed under isotopies. For this purpose we need first the maximal possible causal speed, which, as in the conventional case, is characterized by the null isovector

$$ds^2 = dr^k\, \hat{b}^2\, dr^k - dt\, c_o{}^2 \hat{b}_4{}^2\, dt = 0, \tag{13.55}$$

yielding the following

DEFINITION 13.6 (Santilli (1983a), (1989b)): The maximal, local, causal speed of Santilli's general and restricted isospecial relativities is the maximal speed of a massive particle and/or of a signal verifying the principle of cause and effects, given explicitly by

$$V_{Max} = \left|\frac{dr}{dt}\right|_{Max} = c_o \frac{\hat{b}_4}{\hat{b}} \tag{13.56}$$

The first fundamental postulate of the Santilli's relativities can then be formulated as follows.

POSTULATE I *(loc. cit.): The invariant speed of Santilli's general and restricted isospecial relativities is not given by the speed of light, but by the local, maximal, causal speed for each given physical medium.*

Consider two particles with identical speeds $v_1 = v_2 = c = c_o\, \hat{b}_4/\hat{b}$. Then, their isorelativistic composition yields

$$V_{Tot} = \frac{2c}{1 + \hat{b}} \neq c, \tag{13.57}$$

Consider, on the contrary, two particles with maximal causal speeds, $v_1 = v_2 = V_{Max} = c/\hat{b}$. Then their isorelativistic sum is given by

$$V_{Tot} = \frac{2c/\hat{b}}{2} = V_{Max}, \tag{13.58}$$

thus illustrating the postulate.

Einstein's special relativity is a particular case of Postulate I because it implies $V_{Max} = c = c_o$. Nevertheless, it is remarkable to note that the invariant quantity of the special relativity, strictly speaking, is not c_o but the maximal causal speed.

The plausibility of the above postulate is illustrated by the case of the *Cerenkov light*, where the speed of light in water is $c = c_o/n < c_o$ and the electrons can travel at speeds higher than c, all the way to the limit speed c_I.

Under these conditions, the speed of light c cannot possibly be an invariant of any relativity, irrespective of wether conventional or isotopic. In fact, c varies from medium to medium. Santilli's maximal causal speed is instead an invariant. In fact, since water is homogeneous and isotropic, we have $\hat{b} = \hat{b}_4$ and $V_{Max} = c_o\hat{b}_4/\hat{b} = c_o$. This provides an invariant explanation of the reason why electrons in water can travel at speeds higher than the local speed of light.

The most visible departure of the isospecial from the conventional relativity is given by the following

POSTULATE II *(loc. cit.): The maximal possible, causal speed for Santilli's general and restricted isospecial relativities can be smaller, equal or bigger than the speed of light in vacuum* c_o

$$V_{Max} = c_o \frac{b_4}{\hat{b}} \begin{smallmatrix} > \\ = \\ < \end{smallmatrix} c_o, \tag{13.59}$$

depending on the local physical conditions of the medium considered.

As one can see, Santilli's studies therefore predict the theoretical possibility of "breaking of the barrier" of the speed of light by causal, physical signals. The proposal was originally submitted in Santilli (1982a), and then elaborated in (1983a), (1988c) and (1991d).

A first example can be given via Nielsen-Picek metric (13.27) which yields for the interior of light mesons

$$V_{Max} = c_o \frac{1 + \alpha}{1 - \alpha/3} \tag{13.60}$$

thus characterizing the maximal speed for the interior of pions

$$\alpha \cong -3.8\times10^{-3}, \qquad V_{Ma} = 0.9995\, c_o < c_o, \tag{13.61}$$

while, in the transition to the heavier kaons we have

$$\alpha \cong +0.61\times10^{-3}, \quad V_{Max} \cong (1 + 0.8\times10^{-3})\, c_o > c_o, \tag{13.62}$$

The understanding is that the Nielsen–Picek calculations should be assumed as strictly approximate and valid only at low speed (in fact, they result in a constant isometric). Also, it is predictable that anomalous behaviors will increase with the density of hadrons.

In fact, De Sabbata and Gasperini (1983) conducted explicit calculations within the context of unified gauge theories, and reached the following value for the propagation of a causal signal within the hyperdense medium in the interior of hadrons

$$V_{Max} \approx 75\, c_o . \tag{13.63}$$

The reader is discouraged to approach these novel advances with the mind set on Einstein's conditions due to protracted use. Again, if we have a particle moving in vacuum under action–at–a–distance interactions, we all know that it takes an infinite amount of energy to accelerate the particle to c_o.

We are however considering Santilli's conditions, that is, motion of extended particles within physical media. The reader familiar with the physics of the contact interactions between the particle and the medium knows well that these forces do not admit a potential energy. Yet contact forces are indeed fully capable of producing accelerations. The possibility of passing the speed of light in vacuum under these conditions *without* needing infinite energy is then consequential.

The implications are evidently far reaching in various branches of physics. As an illustration, Postulate II apparently perm,its the achievement of a *true quark confinement* , i.e., a confinement not only with an infinite potential barrier, but also with an identically null probability of tunnel effects into free quarks. In fact, if quarks are physical particles propelled by the hyperdense medium inside hadrons to travel at speeds higher than c_o, they evidently cannot exist as free states in vacuum (Santilli (1989b).

Intriguingly, all modifications of the interior Minkowski metric derived from the phenomenological studies on the behavior of the meanlife with speed appear to confirm Santilli's fundamental hypothesis that the maximal causal speed in the interior of hadrons is bigger than c_o.

More generally, we have the following

PROPOSITION 13.3 (loc. cit.): Any (topology preserving[28]) modification of the Minkowski metric implies a necessary *alteration of the maximal causal speed which can be smaller, equal or bigger than c_o depending on the conditions at hand.*

The reader should be aware that *the quantity c_o is a universal constant for Einstein's special relativity, while the quantity V_{Max} is a local invariant for Santilli's isospecial relativities,* evidently because it can only be defined in the neighborhood of a given point, and it varies from point to point of each given interior medium.

Notice also that the quantity $c = c_o b_4$ is, in general, a geometric quantity and does not necessarily represent a physical speed, evidently because the medium considered can be opaque to all electromagnetic waves, yet permit the motion of particles (see below).

This occurrence can be illustrated via the use, again, of Nielsen-Picek metric (IV.3.21). In fact, we have for kaons

$$c = c_o b_4 = c_o(1 + \alpha) > c_o, \ \alpha > 0, \tag{13.64}$$

The point is that the above value does not necessarily represent the speed of light, because in this classical approximation light cannot propagate inside a hadron (photons and neutrinos do not exist at this classical level). Also note that a given value of $c > c_o$ is not sufficient, per se, to imply that the actual causal speed can indeed be bigger than c_o, because that speed must be computed via quantity (9.5).

Needless to say, the quantity $c = c_o b_4$ can indeed represent the speed of light in particular transparent media, in which case we have

$$b_4 = 1/n, \quad c = c_o/n, \tag{13.65}$$

where n is the index of refraction.

We now pass to the classification of *isofourvectors* in isominkowski space, which can be presented as follows

Isotime-like, when $x^{\hat{2}} < 0,$ (13.66a)

Isonull, when $x^{\hat{2}} = 0,$ (13.66b)

[28] By "topology preserving" we mean any sufficiently smooth and nonsingular modification of the Minkowski metric η which preserves its signature $(+1, +1, +1, -1)$.

Isospace-like, when $\hat{x}^2 > 0.$ (13.66c)

Note that Postulate II is insensitive as to whether the isopoincare transformations are linear or not, and it is instead centrally depends on the inhomogeneity between space and time, i.e., in the differences

$$\hat{b} \neq \hat{b}_4. \tag{13.67}$$

The same situation holds for the mutation of the light cone implies by expressions (13.66). In fact, irrespective of whether the b-quantities are linear or nonlinear (and local or nonlocal) in their variables, we have

$$V_{Max|\,\hat{b} = \hat{b}_4} = c_o \tag{13.68}$$

which is precisely the case of water.

Note that *Santilli's isoparticles traveling faster than* c_o *within physical media are not tachyons* .In different terms, Santilli studies show that it is not sufficient for a particle to travel at speeds bigger than c_o to be a tachyons, because it could be a physical particle verifying the law of cause and effect in interior dynamical conditions. *Thus, to truly have a tachyons, one must have a local speed bigger than the maximal causal speed at the point considered,* whether in the interior or in the exterior problem.

POSTULATE III *(loc. cit.): The dependence of the time intervals with speeds in Santilli's general and restricted isospecial relativities follows the "isotopic time-dilation"*

$$\Delta t = \hat{\gamma}\,\Delta t_o = \Delta t_o \frac{1}{(1-\beta^2)^{\frac{1}{2}}}, \quad \beta^2 = \frac{v^k \hat{b}_k{}^2 v^k}{c_o{}^2 \hat{b}_4{}^2} \tag{13.69}$$

while the dependence of space intervals with speed follows the "isotopic space contraction"

$$\Delta l = \hat{\gamma}^{-1}\,\Delta l_o = \Delta l_o (1-\beta^2)^{\frac{1}{2}}, \tag{13.69}$$

The above postulate appears to be useful for a better understanding of the

stability of nucleus, which is not fully understood at this writing. In fact, this is due to the established instability of the neutron which possesses a mean life of about 15', after which it decays in the familiar form

$$n \Rightarrow p^+ + e^- + \bar{\nu}_e. \qquad (13.69)$$

According to the above evidence, ordinary matter in our environment should be unstable and emit electrons, because of decays (13.69) of the neutron at least in the periphery of nuclei. Numerous regenerative "interpretations" have been submitted in the literature, but they are not fully convincing because, even though plausible for neutrons in the interior of nuclei, they are inapplicable to the neutrons in the periphery of nuclei.

Santilli's isospecial relativities apparently resolve this additional now vexing problem of contemporary physics, because the meanlife of neutron becomes a local quantity which, as such, is dependent on the local physical conditions at hand, and may well increase under nuclear conditions up to the point of full stability of all neutrons within a nucleus.

According to the current nuclear theories, neutrons are assumed to be strictly Einsteinian when member of a nuclear structure. Their meanlife τ then behaves with speeds according to the familiar law

$$\tau = \gamma \tau_o = \tau_o \frac{1}{(1 - v^2/c_o^2)^{\frac{1}{2}}}, \qquad (13.70)$$

But the speeds of nuclear constituents are considerably lower than c_o (in fact, they are known to be much lower than the speeds of the atomic constituents). We can therefore conclude that, except for small relativistic corrections, the mean life of neutrons when members of a nuclear structure remains of the order of 15'. The problem of nuclear stability indicated earlier then follows.

In the transition to Santilli's isospecial relativities the situation is fundamentally different, because now governed by the isotopic law

$$\hat{\tau} = \tau_o \frac{1}{(1 - v\hat{b}^2 v/t c_o^2 \hat{b}_4^2 t)^{\frac{1}{2}}}, \qquad (13.71)$$

An interpretation of the stability of the neutron when members of a nuclear structure would then follow under suitable values of the nuclear interior conditions.

We therefore have the following

PROPOSITION 13.4 (loc. cit.): Any (topology preserving) mutation of the Minkowski metric implies a necessary *alteration of the behavior of the meanlife with speed which can be bigger, equal or smaller than the Einsteinian behavior,*

$$\hat{\tau} = \hat{\gamma}\,\tau_o \begin{array}{c} > \\ = \\ < \end{array} \tau = \gamma\tau_o, \tag{13.72}$$

depending on the local physical conditions of the interior medium considered.

It is hoped the reader begins to see the plausibility of Santilli's relativities. In fact, their central physical idea is that the presence of a physical medium alters the homogeneity and isotropy of the underlying vacuum. If this central hypothesis is correct, it implies a necessary modification of the Minkowski metric. The validity of all postulates of Santilli's relativity is then consequential.

Prior to dismissing the above context because of excessively protracted use of conventional theories, the reader should be aware that numerous phenomenological studies in particle physics result precisely in a modification of the Minkowski metric, as it is the case for Nielsen and Picek (10982), De Sabbata and Gasperini (1983) and numerous others. The above propositions then render Santilli's postulates unavoidable on true scientific grounds.

It is also important to recall that, as proved by Aringazin (1989), *isotopic time-dilation (13.71) is directly universal*, that is, capable of including all possible time dilation laws (universality), in the frame of the experimenter (direct universality).

Thus other modified time dilations existing in the literature have been proved to be a first approximation of Santilli's law (13.71) via one of its many different, possible expansions truncated at a certain power (see the review in Santilli (1991d)).

It should also be noted that, as it is the case for all considerations of this section, *Santilli formulated Postulate .III solely for the behavior of the meanlife of an unstable particle in the INTERIOR dynamical problem, and not to a particle moving in empty space under Einsteinian conditions.*

As a result, *Postulate III is not introduced in this volume for an unstable hadron in a particle accelerator* which, according to the experiments by Grossmann et al. (1987), follows the Einsteinian law.

We pass now to the study of the notion of rest energy of an isoparticle. Consider the fundamental isoinvariant of the theory

$$p^{\hat{2}} = p^k \hat{b}_k^2 p^k - p_4^2 = -m_o^2 c_o^4 \hat{b}_4^4, \tag{13.73}$$

where

$$p = (p^\mu) = mu = (m_o \hat{\gamma} c v, \ m_o \hat{\gamma} c), \tag{13.74}$$

DEFINITION 13.7 (loc. cit.): The energy $\hat{E}$ of an isoparticle on isominkowski spaces $\hat{M}^I(x,\hat{g},\hat{\Re})$ is characterized by the fourth component of the isofourmomentum according to the rule

$$E = p_4, \tag{13.75}$$

and can be expressed in terms of the fundamental isoinvariant (13.73) in the form

$$E^2 = m_o^2 c^4 + p^k b_k^2 p^k, \tag{13.76}$$

We then have the following

POSTULATE IV *(loc. cit.): The rest mass m_o of Santilli's isoparticles on isominkowski spaces $M^I(x,\hat{g},\hat{\Re})$ varies with speed according to the isotopic law*

$$\hat{m} = m_o \hat{\gamma} = \frac{m_o}{(1 - \beta^{\hat{2}})^{\frac{1}{2}}}, \quad \beta^{\hat{2}} = \frac{v^k \hat{b}_k^2 v^k}{c_o \hat{b}_4^2 c_o} \tag{13.77}$$

and the equivalent value of the energy $\hat{E}$ for at rest conditions is given by

$$\hat{E} = m_o c^2 = m_o c_o^2 \hat{b}_4^2(x,p,\mu,\tau,n,....), \tag{13.78}$$

The above postulate has additional far reaching possible implications, ranging from the possible identification of the hadronic constituents with physical particles freely produced in the spontaneous decays, to the problem of the missing mass in the Universe.

Again, this author would like to encourage the reader to move away a mental attitude set for point-like particles in vacuum, and refer instead all Santilli's studies, including the above postulate, to an extended particle inside a physical medium.

PROPOSITION 13.5 (Santilli (1983a), (1989b)): Any (topology preserving) modification of the Minkowski metric implies a necessary alteration of the behavior $\hat{m}$ of the rest mass with energy and of the energy equivalence $\hat{E}$ of the rest mass, which can be bigger, equal or smaller than the corresponding Einsteinian quantities

$$\hat{m} \begin{smallmatrix} > \\ = \\ < \end{smallmatrix} m, \qquad \hat{E} \begin{smallmatrix} > \\ = \\ < \end{smallmatrix} E, \tag{13.79}$$

depending on the local physical conditions of the medium considered.

We now study the redshift under noneinsteinian conditions.

DEFINITION 13.8 (loc. cit.): Santilli's "isoplanewave" is a conventional planewave in Minkowski space $M(x,\eta,\Re)$ under isotopic liftings to isominkowski spaces $\hat{M}^I(x,\hat{g},\hat{\Re})$ with constant isometrics, i.e.,

$$\psi(x) = N e^{i K^\mu \eta_{\mu\nu} x^\nu} \Rightarrow \hat{\psi}(x) = \hat{N} e^{i K^\mu \hat{\eta}_{\mu\nu} x^\nu}, N \in \Re, \ \hat{N} \in \hat{\Re}, \tag{13.80}$$

where K is an isonull vector, i.e.,

$$K^\mu \hat{\eta}_{\mu\nu} K^\nu = 0, \qquad K = (k, \omega/c), \quad \omega/c = 2\pi/\lambda. \tag{13.81}$$

Note that the lifting is here considered solely for the restricted case, because we are treating a global effect of given media (see Fig. 2 of Sect. 11).

Suppose now that such an isoplanewave is detected by two observers S and S', one at rest with respect to the medium, and the other in motion with respect to it, at a relative speed v along the x^3-axis.

As a specific case, the reader may think of ordinary light propagating within our atmosphere which, being transparent, inhomogeneous and anisotropic, is an ideal interior medium for our Lorentz-isotopic relativities. Observer S can be an ordinary observer on our ground, and observer S' can be either moving in the atmosphere or outside it.

Let α be the angle between k and x^3, and let k', ω', and α' be the corresponding quantities for S'.

From the manifest form-invariance of the isoplanewave under the Lorentz-isotopic transformations,

$$K^\mu \hat{\eta}_{\mu\nu} K^\nu = K'^\mu \hat{\eta}_{\mu\nu} K'^\nu, \tag{13.82}$$

it is then easy to see that

$$k'^1 = k^1 = k'^2 = k^2, \tag{13.83a}$$

$$k'^3 = \hat{\gamma}(k^3 - \beta k^4) = |k| \cos\alpha', \ \beta = v/c_o \tag{13.83b}$$

$$k'^4 = \hat{\gamma}(k^4 - \hat{\beta} k^3) = \omega'/c, \ \hat{\beta} = vb/c_o b_4. \tag{13.83c}$$

This leads to the following

POSTULATE V *(loc. cit.): The Doppler's frequency shift for electromagnetic waves propagating within an inhomogeneous and anisotropic physical medium transparent to it (isoplanewave) follows the "isodoppler's laws"*

$$\hat{\omega}' = \omega\hat{\gamma}(1 - \hat{\beta} \cos\alpha), \tag{13.84a}$$

$$\hat{\gamma} = (1 - \hat{\beta}^2)^{-\frac{1}{2}}, \ \hat{\beta}^2 = \frac{vb^2 v}{c_o b_4 c_o^2}, \tag{13.84b}$$

with isotopic aberration

$$\hat{\cos}\,\alpha' = \frac{\cos\alpha - \hat{\beta}}{1 - \hat{\beta}\cos\alpha}, \tag{13.85}$$

The above postulate offers genuine possibilities of resolving another vexing problem of contemporary physics, that of astrophysical bodies currently believed to violate Einsteinian laws under Einsteinian conditions.

In fact, the redshift of certain far distant quasars has recently attained such high values to require the assumption, under the Einsteinian redshift law

$$\omega' = \omega\,\gamma(1 - \beta\cos\alpha), \quad \beta = v / c_o \tag{13.86}$$

that (portions of) quasars travel at speeds higher than c_o, up to speeds of the order of $10c_o$ or more. But the quasars travel in empty space. Thus their center-of-mass trajectory must be strictly Einsteinian in Santilli's views. *The assumption that their speeds is bigger than c_o therefore constitutes a violation of Einsteinian laws under Einsteinian conditions.*

Santilli (1988c) suggested that the quasars redshift could be due in part to the

propagation of light within their hyperdense, inhomogeneous and anisotropic athmospheres. This could reduce the relative speed between the quasars and the associated galaxy, without affecting the current views on the expansions of the universe.

According to Santilli's views, the redshift from far distant quasars, as measured on Earth, could be due to the superposition of:

1) a quantitatively nonignorable isotopic redshift caused by propagation in the quasars' atmospheres;

2) a small isotopic redshift caused by propagation over intergalactic distances for which space is no longer empty, but filled up of radiations, dust, elementary particles, etc.; and

3) a primary Einsteinian redshift caused by the conventional expansion of the Universe.

Mignani (1992) conducted explicit calculations of the above possibilities, by computing preliminary explicit values for the characteristic b-quantities, which apparently confirm the Santilli's predictions.

Mignani essentially assumed, as a first approximation, that quasars are at rest with respect to the associated galaxy, and identified the following expression for the ratio b/b_4

$$B = \frac{b}{b_4} = \frac{(\omega'_1 + 1)^2 - 1}{(\omega'_1 + 1)^2 + 1} \times \frac{(\hat{\omega}'_2 + 1)^2 - 1}{(\hat{\omega}'_2 + 1)^2 + 1}, \qquad (13.87)$$

where ω_1 represents the measured Einsteinian redshift for galaxies and $\hat{\omega}'_2$ represents the redshift for quasars assumed to be mutated. From known astrophysical data, Mignani (*loc. cit.*) then computed the following numerical values of the B-quantities

In summary, according to the above model, the Einsteinian expansion of the Universe is unchanged, because it is that of the Galaxies, but the quasars violation of Einstein's special relativity by speeds higher than c_o in vacuum is eliminated.

Needless to say, values (13.88) are only preliminary and in need of considerable additional studies. Also, several other possibilities remain to be explored, such as the possibility that the quasars are indeed expelled by the galaxies but at Einsteinian speeds.

Despite that, Mignani's results (13.88) have a potentially historical value. In fact, they constitute the first numerical expressions for the characteristic B-quantity of Santilli's interior physical media reached in the literature. As such, they have particular significance, e.g., for the experiments proposed to test Santilli's

isospecial relativities (see Figure 4 below).

GALAXY	QUASAR	B-VALUES
NGC	622UB1	31.91
	BSOI	20.25
NGC 470	68	87.98
	68D	67.21
NGC 1073	BSO1	198.94
	BSO2	109.98
	RSO	176.73
NGC 3842	QSO1	14.51
	QSO2	29.75
NGC 4319	MARK205	12.14
NGC 3067	3C232	82.17

(13.88)

The content of the preceding topic imply the following

PROPOSITION 13.6 (loc. cit.): Any (topology-preserving) modification of the Minkowski metric implies a necessary mutation $\hat{\omega}'$ of the Doppler's redshift which can be bigger, equal or smaller than the Einsteinian value ω'

$$\hat{\omega}' = \omega\,\hat{\gamma}\,(1 - \hat{\beta}\cos\alpha) \quad \gtreqless \quad \omega' = \omega\,\gamma\,(1 - \beta\cos\alpha), \qquad (13.89)$$

depending on the local conditions of the interior medium considered.

It is evident that *Santilli's isospecial relativities relativities, if suitably developed and experimentally confirmed, can indeed provide an infinite family of coverings of Einstein's special relativity, in the sense that:*

A) Santilli's isospecial relativities are constructed with mathematical methods (the Lie–Santilli theory) structurally more general then those of Einstein's special relativity (the conventional Lie's theory);

B) the isospecial relativities represent physical conditions (motion of extended particles within inhomogeneous and anisotropic physical media, deformation of particles, etc.) which are structurally more general than those of Einstein's relativity (point-like particles moving in empty space, etc.); and,

C) the isospecial relativities can approximate Einstein's special relativity as close as desired for $\hat{1}_2 \cong I$, and they all recover by construction Einstein's relativity identically for $\hat{1}_2 \equiv I$.

A visual inspection of Postulates I, II, III, IV and V proves the following important property.

THEOREM 13.5 (loc. cit.): All possible Santilli's isospecial relativities on isospaces $\hat{M}^I(x,\hat{g},\hat{\Re})$ coincide with Einstein's special relativity at the abstract, realization-free level.

Santilli's remarkable achievements are therefore that, despite the generally nonlinear and nonlocal dependence of the various physical quantities (invariant speed, maximal speed, meanlife, rest energy, etc.), Postulates I, II, III, IV and V formally coincide with the corresponding Einsteinian forms at the abstract level.

This illustrate the point made earlier in this section, to the effect that any experimental disproof of the prediction of Santilli's relativities will necessarily demand a revision of the basic postulates of Einstein's special relativity.

Above all, it is remarkable that the "breaking of the barrier" of the speed of light in vacuum by causal signals is ultimately permitted by the very Einsteinian axioms, only realized in a more general way. Finally, in appraising Santilli's relativities, the reader should keep in mind the golden rule of mathematical beauty, which has served throughout the history of science as a solid guidance for physical advances.

In fact, Santilli's isotopic theories provide the most effective methods for the quantitative treatment of dynamics within inhomogeneous and anisotropic media. In case they are disproved by experiments, we remain with the rather elephantiac task of identifying different, but equally effective methods for the treatment of the same problem.

As concluding remarks, let us recall that Einstein's special relativity is based on the following:

<u>PRINCIPLE I:</u> The homogeneity and isotropy of (empty) space;

<u>PRINCIPLE II:</u> The general invariance of the speed of light;

<u>PRINCIPLE III:</u> The general invariance of the physical laws under the broadest possible linear and local group of isometries of the Minkowski space-time;

CLASSICAL EXPERIMENTAL TEST OF SANTILLI'S ISOSPECIAL RELATIVITIES

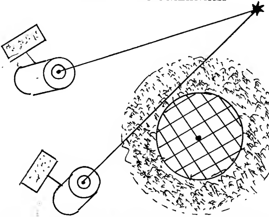

FIGURE 4: A schematic view of the first experiment proposed by Santilli (1988c), (1991d) to test the predictions of his isospecial relativities. As recalled in the text, the isotopies of Einstein's special relativity imply that (nonredshifted) light is redshifted when propagating within an inhomogeneous and anisotropic medium. Santilli therefore proposed the measure of light from a distant star before and after passing through our Earthly atmosphere, or before and after passing through the atmosphere of other objects of our solar system, such as Jupiter. The experiment is fully feasible nowadays, namely, the predictable redshift is fully within the range of our measuring apparata.

To illustrate this point, Santilli (*loc. cit.*) used Mignani's (1992) data (13.88) on quasars's redshift, which yield the following average value for the characteristics B-quantity

$$<|B|> = <|b / b_4|> \approx 84.64 \qquad (a)$$

with corresponding average redshift of the quasars $<|\hat{z}|> \approx 1.25$, while the average redshift of the associated galaxies is given by $<|z|> \approx 0.0125$.

The limit assumption that the quasars are at rest with the associated galaxies then implies that the average redshift caused by the hyperdense quasars atmospheres is given by

$$<|\hat{z}|> - <|z|> \approx 1.24. \qquad (b)$$

Santilli assumes the above average value as proportional; in first approximation to the (average) density of the atmosphere. Then, the assumption that the quasars atmosphere is up to 10^4 denser than our Earth's atmosphere implies the following possible redshift

$$<|\hat{z}|_{\text{Earth}}|> \sim 1.24 \times 10^{-4}, \qquad \text{(c)}$$

which is fully within current experimental capabilities.

A considerable variety of additional experiments can then be conceived. In fact, all historical tests of Einstein's special relativity have been conducted in empty space, as well known. The same experiments can therefore be repeated within physical media [29] (see Santilli (1991d) for details).

from which all other aspects of the relativity can be derived (within inertial reference frames).

But, as stressed by Santilli, *inertial frames are a philosophical abstractions because they do not exist in our Earthly environment, nor they can be attained in our Solar or Galactic systems.* Also, extended particles do not generally move in empty space, but within physical media. The covering principles submitted by Santilli (1983a), (1988c), (1991d) in an attempt to represent more general physical conditions are:

ISOPRINCIPLE I: The inhomogeneity and anisotropy of physical media, with the underlying space remaining homogeneous and isotropic;

ISOPRINCIPLE II: The local invariance of the maximal, speed of causal signals within physical media, with the underlying invariant causal speed in vacuum remaining that of light; and

ISOPRINCIPLE III: The local invariance of physical laws under the most general possible nonlinear and nonlocal groups of isometries of isominkowski spaces of Class I representing physical media, with the conventional linear and local isometries on the Minkowski space being admitted as a particular case;

from which all aspects of the isospecial relativities can be derived, such as the selection, among the multiple infinity of noninertial frames of the Universe, of the

[29] The interested experimenter should always keep in mind the *necessity* of considering inhomogeneous and anisotropic media for a valid test. In fact, as stressed by Santilli, his isodoppler law implies that light passing through water or other homogeneous and isotropic media is not redshifted at all.

subclass of equivalent frames characterized by the general, nonlinear and nonlocal isopoincaré symmetries.

Inertial frames are recovered as a particular case *in first approximation* via the reduction of the general to the restricted isosymmetries (that is, from Fig. 2 of Sect. 11, via the averaging of the characteristic б-functions of the medium considered to b-constants).

Finally, we should mention that the Lorentz-isotopic relativities considered in this section are particular cases of expected, still more general *Lorentz-admissible relativities* for the most general possible open-nonconservative conditions (Santilli (1981a)).

NOTE ADDED IN 1997

Following the appearance of the memoir Santilli (*Foundations of Physics* **27**, 691, 1997) the isogeneral and isospecial relativities are unified into only theory, that of this section, which therefore includes also gravity when the isominkowskian metric is equal to the Riemannian metric.

This unified formulation of the special and general relativities for both exterior and interior problems is requested not only by the lack of invariance of the basic units of space and time for geometries with non-null metrics (see the note added at the end of the preceding section (p. 176), but also by other independent needs.

For instance, curvature is a major obstacle in the inclusion of gravitation in unified gauge theories of electroweak interactions. The formulation of gravitation in a form axiomatically equivalent to that of the electroweak interactions, that is, flat, provides serious grounds for a grand unified theories (loc. cit.).

Similarly, curvature is a major obstacle for a physically consistent operator form of gravity. In fact, current quantum theories are nonunitary, thus having a number of physical shortcomings, such as (loc. cit.): lack of invariance of the basic units of space and time; lack of conservation of Hermiticity in time; lack of invariant numerical predictions; etc.

Santilli's formulation of gravity via his isospecial relativity offers again solid scientific grounds for a resolution of the latter problems too. In fact they permit the construction of an axiomatically consistent operator form of gravity obeying the axioms of the conventional relativistic quantum mechanics (loc. cit.).

I.14: ISOGALILEAN RELATIVITY

In our reverse order of presentation, we now review *Santilli's isogalilean relativities* for interior dynamical conditions, as a particular case of Santilli's isospecial relativities.

More specifically, we shall present the isogalilean relativities as an isogroup contraction of the isospecial relativities. In this way, the content of this section will be a second necessary complement to the gravitational studies of Sect. 12, because valid in their local tangent places in a nonrelativistic approximation.

Santilli's proposed the isotopic generalization of Galilei's relativity in his historical memoir of 1978a, as a particular case of his more general relativity of Lie–admissible type. The generalized relativity was then studied in details, for the nonlinear and nonhamiltonian but local systems, in the two monographs Santilli (1978e) and (1982a) under the name of *Galilei–isotopic relativities*. The generalized relativities were then extended to nonlocal systems in Santilli (1988a) and reached their final formulation in the monographs Santilli (1991c, d).

Again, as it had been the case for the generalization of the special relativity, Santilli constructed his isogalilean relativities alone. In fact, rather oddly, no independent researcher cared to write a paper on such a manifestly basic advancement during its construction, despite the appearance of the monograph Santilli (1982a) in the prestigious series "Text and Monographs in Physics" by Springer–Verlag with the title of Chapter VI, p. 199, *"Generalization of Galilei's Relativity"*.

As of this writing, the *only* paper on the isogalilean relativities that has appeared in print by independent researchers, to this author's best knowledge, is that by Jannussis, Miatovic and Veljanosky (1991) on explicit examples of the new relativities.

To begin, let us identify the *nonrelativistic limit of isominkowski spaces* $\hat{M}^I(x,\hat{g},\hat{\mathfrak{R}})$, which was studied first in Santilli (1988c), Appendix A.

Consider the fundamental isoinvariant of $\hat{M}^I(x,\hat{g},\hat{\mathfrak{R}})$ in the form

$$\frac{1}{c_0^2}\, r^k\, b_k^{\ 2} - t b_4^{\ 2} t = -\frac{R^2}{c_0^2}, \tag{14.1}$$

then it is easy to see that at the limit

$$\epsilon = \frac{R}{c_0} \Rightarrow 0, \tag{14.2}$$

we have the contraction

$$\hat{M}^I(x,\hat{g},\hat{\mathfrak{R}})_{|R/c_o \Rightarrow 0} \Rightarrow \hat{\mathfrak{R}}_t \times \hat{E}(r,\hat{G},\hat{\mathfrak{R}}), \tag{14.3}$$

namely, the isominkowski spaces contract into Santilli's isoeuclidean spaces for the space coordinates

$$\hat{E}(r,\hat{G},\hat{\mathfrak{R}}): \; r^{\hat{2}} = r^t \, \hat{G} \, r, \tag{14.4a}$$

$$\hat{G} = \mathrm{diag.}\,(\hat{b}_1{}^2, \hat{b}_2{}^2, \hat{b}_3{}^2) > 0, \tag{14.4b}$$

$$\hat{\mathfrak{R}} = \mathfrak{R}\hat{1}_2, \; \hat{1}_2 = \mathrm{diag.}\,(\hat{G}^{-1}, \hat{G}^{-1}), \tag{14.4c}$$

multiplied the isospace for time

$$t^{\hat{2}} = t \, \hat{b}_4{}^2 \, t \Rightarrow \hat{\mathfrak{R}}_t = \mathfrak{R}\hat{1}_t, \; \hat{1}_t = \hat{b}_4{}^{-2}, \tag{14.5}$$

Thus,

$$\hat{M}^I(x,\hat{g},\hat{\mathfrak{R}})\,|_{R/c_o \Rightarrow 0} = \hat{\mathfrak{R}}_t \times \hat{E}(r,\hat{\delta},\hat{\mathfrak{R}}), \tag{14.6}$$

which is a clear isotopy of the conventional contraction

$$N(x,\eta,\mathfrak{R})_{|R/c_o \Rightarrow 0} \Rightarrow \mathfrak{R}_t \times E(r,\delta,\mathfrak{R}). \tag{14.7}$$

Remarkably, Santilli (1988a) first identified isospace $\hat{\mathfrak{R}}_t \times \hat{E}(r,\hat{\delta},\hat{\mathfrak{R}})$ as the fundamental space of his isogalilean relativities, and then proved its compatibility with the isominkowski soace in (1988c).

We now review the nonrelativistic limit of the isopoincaré symmetries $\hat{P}(3.1)$. We shall use the isotopic generalization of the conventional techniques of group contraction of the Poincaré group P(3.1) into the Galilei's group G(3.1) (see, e.g., Gilmore (1974))

$$P(3.1) = \{O(3.1) \otimes T(3.1)\}\,|_{R/c_o \Rightarrow 0} = G(3.1) = [O(3) \otimes T(3)] \times [T(3) \times T(1)] \tag{14.8}$$

which was studied for the first time also in Santilli (1988c), Appendix A, via the following

THEOREM 14.1 (Isotopic Inönü–Wigner contractions, (loc. cit.)): Let $\hat{\mathfrak{g}}$ *be a (finite-dimensional) Lie–Santilli algebra defined on an isotopic field* $\hat{F}$ *of real or complex*

numbers, and consider its direct-sum decomposition as isovector space

$$\hat{g} = \hat{g}_0 \oplus \hat{g}_1, \tag{14.9}$$

Let $\hat{U}(\epsilon)$ be an isotransformation on $\hat{g}$ which becomes singular at the limit $\epsilon \Rightarrow 0$, and which is such that

$$\hat{U}(0) * \hat{g}_0 = \hat{g}_0, \tag{14.10a}$$

$$\hat{U}(0) * \hat{g}_1 = 0. \tag{14.10b}$$

Then $\hat{g}$ can be contracted with respect to $\hat{g}_0$ into a new isoalgebra $\hat{g}'$ iff $\hat{g}_0$ is a closed subgroup of $\hat{g}$, in which case:
1) g_0 is a subalgebra of both $\hat{g}$ and $\hat{g}'$;
2) $\hat{g}'_1$ becomes an Abelian invariant subalgebra of $\hat{g}'$; and
3) $\hat{g}$ is non-semisimple.

The application of the above theorem to the isopoincaré algebra $\hat{P}(3.1)$ is straightforward. Consider the basis of $\hat{P}(3.1)$; decompose it as an isovector space in the form

$$\hat{P}(3.1) = \hat{g}_0 \oplus \hat{g}_1 = (J_{ij} + P_k + P_4) \oplus J_{k4}; \tag{14.11}$$

redefine it in the vicinity of the "north pole" (0,R) (see Gilmore (*loc. cit.*) p. 451), and perform the contractions

$$J_k = \mathrm{Lim}_{|R/c_o \Rightarrow 0}\, \epsilon_{kij} J_{ij} = \epsilon_{kij}(r_i p_j - r_j p_i), \tag{14.12a}$$

$$P_k = \mathrm{Lim}_{|R/c_o \Rightarrow 0}\, p_k = p_k, \quad H = \mathrm{Lim}_{|R/c_o \Rightarrow 0}\, p_4 = p_4 = E, \tag{14.12b}$$

$$G_k = \mathrm{Lim}_{|R/c_o \Rightarrow 0}\, J_{k4}/R = \mathrm{Lim}_{R \Rightarrow 0}(x_k p^4 - x^4 p_k)/R, \tag{14.12c}$$

$$i, j, k = 1, 2, 3,$$

where we have assumed a new nonrelativistic expression for the energy.

Then, it is easy to see that the isocommutation rules of $\hat{P}(3.1)$, Eq.s (13.33), are contracted into the isocommutation rules of the Galilei-isotopic algebras $\hat{G}(3.1)$, Eq.s (14.30) below, in exactly the same way as the commutation rules of $P(3.1)$ contract into those of $G(3.1)$.

The isotopic liftings of contraction (14.8) then identifies the structure of the isogalilean group as follows

$$\hat{P}(3.1) = \{\hat{O}(3.1) \otimes \hat{T}(3.1)\}_{|R/c_o \Rightarrow 0} = \hat{G}(3.1) = [\hat{O}(3) \otimes \hat{T}(3)] \times [\hat{T}(3) \times \hat{T}(1)]. \quad (14.13)$$

The crucial local isomorphism between all infinitely possible isotopes $\hat{G}(3.1)$ and the conventional group G(3.1) can therefore be inferred already from these introductory remarks.

We now review Santilli's construction of $\hat{G}(3.1)$. As well known, the conventional *Galilei symmetry* G(3.1) (see, e.g., Levy-Leblond (1971) or Sudashan and Mukunda (1974)) can be defined as the largest Lie group of *linear and local* transformations leaving invariant the separations

$$t_a - t_b = \text{inv.},$$

$$(r_{ia} - r_{ib})\,\delta_{ij}\,(r_{ja} - r_{jb}) = \text{inv. at } t_a = t_b, \quad (14.14)$$

$$i, j = 1, 2, 3 \ (= x, y, z), \qquad a = 1, 2, \ldots, N$$

in $\Re_t \times T^*E(r,\delta,\Re)$, where $\Re_t$ represents time, $E(r,\delta,\Re)$ is the conventional Euclidean space, and T^*E its cotangent bundle (phase space), with metric $\delta = \text{diag.}(1,1,1)$ over the reals $\Re$.

The explicit form of the celebrated *Galilei transformations* is given by

$$t' = t + t^\circ, \quad \text{translations in time} \quad (14.15a)$$

$$r'_{ia} = r_{ia} + r^\circ_i, \quad \text{translations in space} \quad (14.15b)$$

$$r'_{ia} = r_{ia} + t^\circ v^\circ_i, \quad \text{Galilei bosts} \quad (14.15c)$$

$$r'_a = R(\theta)\, r_a, \quad \text{rotations.} \quad (14.15d)$$

A classical realization of G(3.1) for a system of N particles with non-null masses, $m_a \neq 0$, $a = 1, 2, \ldots, N$ herein assumed, is characterized by the (ordered sets of) parameters

$$w = (w_k) = (\theta_i, v^\circ_i, r^\circ_i, t^\circ), \quad k = 1,2,\ldots 10, \quad i = 1, 2, 3 \quad (14.16)$$

and generators

$$X = (X_k) = (J_i, G_i, P_i, H), \tag{14.17a}$$

$$J_i = \sum_a \epsilon_{ilm} r_{la} p_{ma}, \quad P_i = \sum_a p_{ia}, \tag{14.17b}$$

$$G_i = \sum_a (m_a r_{ia} - tp_{ia}), \quad H = p_{ia} p_{ia} / 2m_a + V(r_{ab}), \tag{14.17c}$$

$$r_{ab} = r_a - r_b, \quad i = 1, 2, 3, \quad k = 1, 2, ..., 10, \quad a, b = 1, 2, ..., N$$

with canonical realization of the Lie algebra G(3.1) via the conventional Poisson brackets

$$G(3.1): \quad [J_i, J_j] = \epsilon_{ijk} J_k, \quad [J_i, P_j] = \epsilon_{ijk} P_k, \tag{14.18a}$$

$$[J_i, G_j] = \epsilon_{ijk} G_k, \quad [J_i, H] = 0, \tag{14.18b}$$

$$[G_i, P_j] = \delta_{ij} M, \qquad [G_i, H] = P_i, \tag{14.18c}$$

$$[P_i, P_j] = [G_i, G_j] = [P_i, H] = 0, \tag{14.18d}$$

$$M = \sum_a m_a, \tag{14.18e}$$

Casimir invariants

$$C^{(o)} = 1, \qquad C^{(1)} = P^2 - 2MH, \quad C^{(2)} = (MJ - G \wedge P)^2, \tag{14.19}$$

and canonical realization of the group structure

$$G(3.1): \quad a' = g(w)\, a = \{ \exp [w_k \, \omega^{\mu\nu} (\partial_\nu X_k)(\partial_\mu)]\} \, a \tag{14.20}$$

$$\partial_\mu = \partial / \partial a^\mu, \quad a = (a^\mu) = (r_{ia}, p_{ia}), \; \mu = 1,2,...,6N,$$

where $\omega^{\mu\nu}$ is the canonical Lie tensor (Sect. 9).

As now familiar, the starting ground of the liftings $G(3.1) \Rightarrow \hat{G}(3.1)$ is the infinite number of isotopes $\hat{E}(r,\hat{\delta},\hat{\Re})$ of the Euclidean space $E(r,\delta,\Re)$ which are extended to the isocotangent bundle $T^*\hat{E}(r,\hat{\delta},\hat{\Re})$. A nonhamiltonian system of N particles, Eq.s (1.1), is then introduced in such isospace with the familiar local coordinates $a = (a^\mu) = (r_{ka}, p_{ka})$, $\mu = 1, 2, ..., 6N$, $k = 1, 2, 3$ (= x, y, z), and $a = 1, 2, ..., N$.

The system is then represented via the Pfaffian–Santilli variational principle (Sect. 7) which is essentially based on the one-isoform on $T^*\hat{E}_1(r,\hat{\delta},\hat{\mathfrak{R}})$ (Sect. 9)

$$\Phi_I = \theta_I \times \hat{T}_I = R^\circ_\mu \hat{T}_I{}^{\mu\nu}\, \partial a^\nu, \tag{14.21a}$$

$$R^\circ = (p, 0), \quad \hat{T}_I = \mathrm{diag.}\,(\hat{\delta}, \hat{\delta}), \tag{114.21b}$$

$$\hat{\delta} = \mathrm{diag.}\,(b_1{}^2, b_2{}^2, b_3{}^2), \; b_k = b_k(t, r, p, \dot{p}, \ldots) > 0. \tag{14.21c}$$

The isospaces for the treatment of the symmetries of the systems are $T^*\hat{E}_2(\hat{\delta},\hat{\mathfrak{R}}) = T^*\hat{E}(r,\hat{G},\hat{\mathfrak{R}})$, which are characterized by the two-isoform

$$\hat{\Omega}_2 = [\tfrac{1}{2}\, \hat{T}_{2\mu}{}^{\alpha}(a\ldots)\, \omega_{\alpha\nu}]\, \partial a^\mu \wedge \partial a^\nu = d[\hat{T}_1(a,..)\times R^\circ]_\mu\, da^\mu, \tag{14.22a}$$

$$\hat{T}_2 = \mathrm{Diag.}\,(\hat{G}, \hat{G}), \quad \hat{G} = \mathrm{diag.}\,(B_1{}^2, B_2{}^2, B_3{}^2), \; B_k > 0, \tag{14.22b}$$

where the methods to construct the isometrics $\hat{G}$ from $\hat{\delta}$ are assumed to be known (see Sect. 9).

The Lie–Santilli brackets characterized by two-forms (13.22) are given by the now familiar expression in $T^*\hat{E}(r,\hat{G},\hat{\mathfrak{R}})$

$$[A \hat{,} B] = \frac{\partial A}{\partial a^\mu}\, \omega^{\mu\alpha}\, \hat{1}_{2\alpha}{}^{\nu}\, \frac{\partial B}{\partial a^\nu}, \tag{14.23a}$$

$$\hat{1}_2 = \mathrm{diag.}\,(\hat{G}^{-1}, \hat{G}^{-1}), \tag{14.23b}$$

where $\omega^{\mu\alpha}$ is the familiar canonical Lie tensor.

We are now equipped to introduce the following

DEFINITION 14.1 (loc. cit.): Santilli's "general, nonlinear and nonlocal, isogalilean symmetries" $\hat{G}(3.1)$ are given by the Lie–Santilli groups of the most general possible transformations on $\hat{\mathfrak{R}}_t \times T^*\hat{E}(r,\hat{G},\hat{\mathfrak{R}})$

$$t_a - t_b = \mathrm{inv.}, \tag{14.24a}$$

$$(r_{ka} - r_{kb})\, B_k{}^2(t, r, p, \ldots)\, (r_{ka} - r_{kb}) = \mathrm{inv.} \quad \text{at } t_a = t_b, \tag{14.24b}$$

$$t_a, t_b \in \hat{\Re}_t, \qquad r_a, r_b \in T^*\hat{E}(r,\hat{G},\hat{\Re}) \tag{14.24c}$$

where $\hat{\Re}_t$ *is an isotopic lifting of the conventional field* $\Re_t$, *called Santilli's "isotime field", with explicit structure*

$$\hat{\Re}_t = \Re \hat{1}_t, \quad \hat{1}_t = B_4^{-2}(t, r, p,...), \qquad B_4 > 0, \tag{14.25}$$

$T^*\hat{E}(r,\hat{G},\hat{\Re})$ *is the isocotangent bundle for isosymplectic two-isoforms with isometrics (14.22), and the four functions* B_1, B_2, B_3, *and* B_4, *besides being independent and positive-definite, are arbitrary nonlinear and nonlocal (e.g., integral) functions on all possible, or otherwise needed local variables and quantities.*

We now reproduce without proof the following

THEOREM 14.1 (loc. cit.): Santilli's general, nonlinear and nonlocal, classical realization of the Galilei-isotopic symmetries $\hat{G}(3.1)$ *on* $\hat{\Re}_t \times T^*\hat{E}(r,\hat{G},\hat{\Re})$ *as per Definition 14.1, can be written*

$$t' = t + t^\circ \tilde{B}_4^{-2}, \qquad \text{iso-time translations} \tag{14.26a}$$

$$r_i' = r_i + r^\circ_i \tilde{B}_i^{-2}, \qquad \text{iso-space translations} \tag{14.26b}$$

$$r_i' = r_i + t^\circ v^\circ_i \tilde{B}_i^{-2}, \quad \text{iso-Galilei boosts} \tag{14.26c}$$

$$r' = \hat{R}(\theta) * r, \qquad \text{isorotations,} \tag{14.26d}$$

where the $\tilde{B}$*-functions are generally nonlinear and nonlocal in all possible local variables and quantities to be identified shortly. Moreover, the Galilei-isotopic symmetries* $\hat{G}(3.1)$ *are characterized by the Lie-isotopic brackets (14.23a), with explicit form*

$$[A\hat{,} B] = \frac{\partial A}{\partial r_{ka}} B_k^{-2} \frac{\partial B}{\partial p_{ka}} - \frac{\partial A}{\partial p_{ka}} B_k^{-2} \frac{\partial B}{\partial r_{ka}}, \tag{14.27}$$

and possess the following structure:

1) the conventional Galilean parameters

$$w = (w_k) = (\theta_i, r^\circ_i, v^\circ_i, t^\circ), \quad k = 1,2,...10, \tag{14.28}$$

and the conventional Galilean generators, but now defined on isospace $\hat{\mathfrak{R}}_t \times T^*\hat{E}(r,G,\hat{\mathfrak{R}})$, i.e.,

$$J_i = \sum_a \epsilon_{ijk} r_{ja} p_{ka}, \quad P_i = \sum_a p_{ia}, \tag{14.29a}$$

$$G_i = \sum_a (m_a r_{ia} - t p_{ia}), \tag{14.29b}$$

$$H = p_{ka} B_k^{\;2} p_{ka} / 2m_a + V(r_{ab}), \tag{14.29c}$$

$$r_{ab} = |r_a - r_b|^{\hat{\frac{1}{2}}} = \{(r_{ka} - r_{kb}) B_k^{\;2} (r_{ka} - r_{kb})\}^{\frac{1}{2}}, \tag{14.29d}$$

2) the Lie–Santilli algebra

$$\hat{G}(3.1): \; [J_i \hat{,} J_j] = \epsilon_{ijk} B_k^{-2} J_k, \quad [J_i \hat{,} P_j] = \epsilon_{ijk} B_j^{-2} P_k, \tag{14.30a}$$

$$[J_i \hat{,} G_j] = \epsilon_{ijk} B_j^{-2} G_k, \quad [J_i \hat{,} B] = 0, \tag{14.30b}$$

$$[G_i \hat{,} P_j] = \delta_{ij} M B_j^{-2}, \quad [G_i \hat{,} B] = 0, \tag{14.30c}$$

$$[P_i \hat{,} P_j] = [G_i \hat{,} G_j] = [P_i \hat{,} B] = 0, \tag{14.30d}$$

3) the Lie–Santilli group

$$\hat{G}(3.1): \quad r' = \{ [\textstyle\prod_k e_{|\xi}^{\,w_k \,\omega^{\mu\sigma} \times I_2^{\;\sigma\nu} (\partial_\nu X_k)(\partial_\mu)}] \hat{1}_2 \} * r, \tag{514.31}$$

4) the local isocasimir invariants

$$\hat{C}^{(0)} = \hat{1}_2, \quad \hat{C}^{(1)} = (P\hat{G}P - MH)\hat{1}_2, \tag{14.31a}$$

$$\hat{C}^{(2)} = (MJ - G \wedge P)^{\hat{2}} = \{ (MJ - G\wedge P)\hat{G}(MJ - G\wedge P) \} \hat{1}_2, \tag{14.31b}$$

5) the explicit expressions of the $\tilde{B}_i$ functions

$$\tilde{B}_i^{-2}(r^\circ) = B_i^{-2} + r^\circ_j [B_i^{-2} \hat{,} P_j] / 2! + r^\circ_m r^\circ_n [[B_i^{-2} \hat{,} P_m] \hat{,} P_n] / 3! + \dots \tag{14.32a}$$

$$\tilde{B}_i^{-2}(v^\circ) = B_i^{-2} + v^\circ_j [B_i^{-2} \hat{,} G_j] / 2! + v^\circ_m v^\circ_n [[B_i^{-2} \hat{,} G_m] \hat{,} G_n] + \dots \tag{14.32b}$$

while $\tilde{B}_4^{-2}(t^\circ)$ is the solution of the algebraic equation

$$r(t + t^\circ \tilde{B}_4^{-2}) = \{ e_{|\xi}^{t^\circ \omega^{\mu\sigma} \times 1_2^{\sigma\nu} (\partial_\mu H)(\partial_\nu)} \} r. \tag{14.33}$$

The isogalilean symmetries so constructed result to be all locally isomorphic to the conventional Galilei symmetry under the conditions of sufficient smoothness, nonsingularity and positive-definitness of the isounits. Finally, all isosymmetries $\hat{G}(3.1)$ can approximate the conventional symmetry $G(3.1)$ as close as desired whenever the isounits approach the conventional unit, and they all admit the conventional symmetry as a particular case by construction.

The preceding results evidently include as a particular case the characterization of the isoeuclidean symmetries $\hat{E}(3) = \hat{O}(3) \otimes \hat{T}(3)$, as well as of the isorotational symmetries $\hat{O}(3)$ of the next section.

It is an instructive exercise for the interested reader to prove that the infinite family of isosymmetries $\hat{G}(3.1)$ so constructed do indeed verify the conditions of Definition 14.1 and, in particular, do constitute isosymmetries of invariants (14.24).

COROLLARY 14.1.1: In the particular case of constant isometrics $\hat{\delta}$, we have

$$\mathfrak{R}_t \times T^*\hat{E}_1(r,\hat{\delta},\hat{\mathfrak{R}}) \equiv \hat{R}_t \times T^*\hat{R}_2(r,G,\hat{\mathfrak{R}}) \equiv \mathfrak{R}_t \times T^*\hat{E}(r,\hat{\delta},\hat{\mathfrak{R}}), \tag{13.344a}$$

$$g = G = \text{diag.}(b_1^2, b_2^2, b_3^2), \quad b_k = \text{const.} > 0; \tag{14.34b}$$

$$\hat{1}_t = b_4^{-2} = \text{const.} > 0, \tag{14.34c}$$

the $\tilde{B}$-quantities coincide with the diagonal elements of the isounits,

$$\tilde{B}_i^{-2}(r^\circ) \equiv \tilde{B}_i^{-2}(v^\circ) \equiv B_i^{-2} \equiv b_i^{-2}, \quad \tilde{B}_4^{-2}(t^\circ) \equiv b_4^{-2}. \tag{14.35}$$

and the general isogalilean transformations become the linear and local isogalilean transformations

$$t' = t + t^\circ b_4^{-2}, \tag{14.36a}$$

$$r'_i = r_i + r^\circ_i b_i^{-2}, \tag{14.36b}$$

$$r'_i = r_i + t^\circ v^\circ_i b_i^{-2}, \tag{14.36c}$$

$$r' = \hat{R}(\theta) * r. \tag{14.36d}$$

called Santilli's "restricted isogalilean transformations".

The latter isotransformations have important implications from a relativity viewpoint., because they imply the possibility of preserving the inertial frames.

We now pass to the review of Santilli's isogalilean relativities. As well known, the conventional *Galilei relativity* (see, again, Levy-Leblond (1971) or Sudarshan and Mukunda (1974)) is a form-invariant description of physical systems under the Galilei's symmetry G(3.1) or, equivalently, under Galilei's transformations (14.15)

The relativity is verified in our physical reality only for a rather small class of Newtonian systems, called by Santilli (1982a) *closed selfadjoint systems.* These are systems (such as our planetary system) which verify the conventional total Galilean conservation laws when isolated, and can only admit internal forces of local (differential) and potential (selfadjoint) type without collisions.

For all remaining Newtonian systems, Galilei's symmetry is violated according to a number of mechanisms classified by Santilli (1982a), pp. 344-348, into

isotopic, selfadjoint, semicanonical, canonical and essentially selfadjoint breaking.

In the final analysis, the limitations of Galilei's relativity are inherent in its mathematical structure. In fact,

1) The *linear* character of Galilei's transformations is at variance with the generally non*linear* structure of the systems of the physical reality of the interior dynamical problem, as established by incontrovertible evidence;

2) The *local* (differential) character of Galilei's relativity is at variance with the generally non*local* (integral) nature of the systems in our Earthly environment;

3) The strictly *Hamiltonian* (canonical) structure of Galilei's relativity is at variance with the generally non*hamiltonian* character of physical systems of our reality;

and other reasons.

An infinite family of isotopic generalizations of the Galilei symmetry, under the name of *Santilli's isogalilean symmetries* $\hat{G}(3.1)$ has been reviewed in this section. Their main features are the following:

A) The isogalilean symmetries characterize the more general class of *closed non-selfadjoint systems*. These are systems (such as Jupiter) which verify the conventional, total, Galilean conservation laws when isolated, while admitting the additional class of nonlocal, nonhamiltonian and nonnewtonian internal forces as in Eq.s (1.1).

B) The isogalilean symmetries possess structure (14.13), and result to be locally isomorphic to the conventional symmetry G(3.1) under the positive-definiteness of the underlying isounits, by admitting the latter as a particular case. In this sense, $\hat{G}(3.1)$ provides an infinite family of *Lie-Santilli coverings* of G(3.1).

C) All symmetries $\hat{G}(3.1)$ can be explicitly constructed via the Lie-Santilli theory, that is, via the use of the same parameters and generators (conserved quantities) of the conventional symmetry, but via the most general possible, axiom-preserving realizations of Lie-Santilli algebras and groups. In this way, an infinite number of symmetries $\hat{G}(3.1)$ can be constructed for each given Hamiltonian $H = T + V$ (i.e., for each given potential-selfadjoint forces), as characterized by an infinite number of possible interior physical media.

DEFINITION 14.2:: Santilli's "general, nonlinear and nonlocal, isogalilean relativities" are the a form-invariant description of physical systems under the isogalilean symmetries $\hat{G}(3.1)$ on *isospaces* $\hat{\mathfrak{R}}_t \times T^*\hat{E}(r,\hat{G},\hat{\mathfrak{R}})$, $\hat{\mathfrak{R}} = \mathfrak{R}\,\hat{1}_2$ = diag. $(\hat{G}^{-1}, \hat{G}^{-1})$,, $\hat{\mathfrak{R}}_t = \mathfrak{R}\,\hat{1}_t$, $\hat{G} > 0$, $\hat{1}_2 > 0$, $\hat{1}_t > 0$, *with corresponding, infinite family of general isogalilean transformations (14.26). Santilli's "restricted isogalilean relativities"* occur *for the linear and local subclass of isotransformations (14.36).*

To begin the understanding of Santilli's isogalilean relativities, one should keep in mind that the isotopic formulations were conceived and constructed by

Santilli in such a way to coincide with the original formulation at the abstract, realization-free level. The reader should therefore expect that *the isogalilean relativities coincide, by construction, with the conventional relativity at the abstract, coordinate-free level.*

As reviewed earlier, *the generalized physical laws characterized by the isogalilean relativities are given by the modification of the Galilean laws caused by the addition of contact, nonlinear, nonlocal and nonhamiltonian forces between the particles and the interior physical medium.*

The best illustration was given in Fig. 2, Sect. 12, via the generalization of the celebrated *Galilean law for the uniform motion in vacuum*

$$r'_i = r_i + t^\circ v^\circ_i, \tag{14.37a}$$

$$p'_i = p_i + m v^\circ_i, \tag{14.37b}$$

into *Santilli's isoboosts*

$$r'_i = r_i + t^\circ v^\circ_i \tilde{B}_i^{-2}(t, r, p, ...), \tag{14.38a}$$

$$p'_i = p_i + m v^\circ_i \tilde{B}_i^{-2}(t, r, p,. \tag{14.38b}$$

which represent precisely the behavior of an extended particle with original speed v° in "free motion" within a physical medium, that is, when all potential forces are null, but the particle is subject to the contact forces with the medium.

One can then see that, while the original law (14.37) describes the uniform motion in vacuum, Santilli's covering law (14.38) describes a decelerated or accelerated motion, depending on the physical conditions at hand.

A fully similar situation occurs for all other physical laws. The case of the generalized rotations is reviewed in the next section.

To understand the equivalence of Galilean law with Santilli's coverings, we note that, despite the nonlinearity and nonlocality of the latter, *all isogalilean transformations (14.38) locally coincide with the conventional transformations (14.37),* i.e., they coincide at a given, fixed value $\bar{t}, \bar{r}, \bar{p}$, ... of the local variables

$$t^\circ \tilde{B}_i^{-2}(\bar{t}, \bar{r}, \bar{p}, ...) \equiv t'^\circ = \text{cost.}, \qquad r^\circ \tilde{B}_i^{-2}(\bar{t}, \bar{r}, \bar{p}, ...) \equiv r'^\circ, \tag{14.39a}$$

$$v^\circ \tilde{B}_i^{-2}(t, \bar{r}, \bar{p}, ...) \equiv v'^\circ = \text{const.}, \qquad \hat{R}(\theta)_{\bar{r}, \bar{p}, ...} \equiv R(\theta'). \tag{14.39b}$$

A similar situation evidently holds for the physical laws, namely, we can state that *the physical laws characterized by Santilli's isogalilean relativities are locally equivalent to the laws of the conventional Galilei relativity.*

Second, we recall that the topological structure of Galilei's law (14.37) is given by

$$T(v^\circ)\, r_i = r_i - t^\circ v^\circ_i, \quad T(v^\circ)\, p_i = p_i - m v^\circ_i, \tag{14.40a}$$

$$T(v^\circ) = e_{|\xi}^{\, v^\circ_j \omega^{\mu\nu} (\partial_\nu G_j)(\partial_\mu)}, \tag{14.40b}$$

$$T(v^\circ)\, T(v^\circ) = T(v^\circ + v^{\circ\prime}), \tag{14.40c}$$

while the corresponding structure of Santilli's covering law (14.38) is given by

$$\hat{T}(v^\circ) * r_i = r_i - t^\circ v^\circ_i \tilde{B}_i^{-2}, \quad \hat{T}(v^\circ) * p_i = p_i - m v^\circ_i \tilde{B}_i^{-2}, \tag{14.41a}$$

$$\hat{T}(v^\circ) = \{ [e_{|\xi}^{\, v^\circ_j \omega^{\mu\sigma} \hat{1}_2{}^{\sigma\nu} (\partial_\nu G_j)(\partial_\mu)}] \hat{1}_2 \}. \tag{14.41b}$$

$$\hat{T}(v^\circ) * \hat{T}(v^{\circ\prime}) = \hat{T}(v^\circ + v^{\circ\prime}). \tag{14.41c}$$

The identity of laws (14.37) and (14.38) at abstract, realization-free level is then evident.

However, the above local and global equivalences are still insufficient for a true understand of Santilli's isogalilean relativities.

It is at this point were our reversed order of presentation acquires its full light. In fact, Santilli's primary and most important conception of his generalized relativities is that of novel *GEOMETRIC* character.

In fact, the understanding of the physical equivalence of laws (14.37) and (14.38) requires the knowledge that,

in the same way as the Galilean uniform motion (14.37) is a geodesic in Euclidean space $E(r,\delta,\Re)$, Santilli's isouniform motion (14.38) is an isogeodesic in the isoeuclidean spaces $\hat{E}(r,\hat{\delta},\hat{\Re})$ (see *Fig. 2 of Sect. 11 for more details*).

In this way, and only in this way, the reader can understand that Santilli constructed his coverings of the Galilei relativity in such a way to preserve the underlying axioms, and realized them via more general geometries.

The reader interested in acquiring a technical knowledge of Santilli's

isogalilean relativities is therefore suggested to study them via a particularization of the isoriemannian geometry of Sect.s 11 and 12 for null isoconnection coefficients.

We have reached in this way the most important physical result of this section, which can be expressed as follows:

THEOREM 14.2 (loc. cit.): All possible Santilli's isogalilean relativities on $\hat{R}_t \times T^*\hat{E}(r,\hat{G},\hat{\Re})$ *coincide with the conventional Galilei relativity on* $\Re_t \times T^*E(r,\delta,\Re)$ *at the abstract, realization-free level, that is. at the abstract limit considered:*
a) all infinitely possible isogalilean symmetries $\hat{G}(3.1)$ *coincide with the conventional Galilei symmetry* $G(3.1)$ *(global aspect);*
b) all infinitely possible isogalilean transformations (14.26) locally coincide with the conventional transformations (14.15) (local profile); and
c) all physical laws of Santilli's isogalilean relativities coincide, by construction, with the conventional Galilean laws (physical profile).

Despite such mathematical and physical unity, the differences between Santilli's and Galilei's relativities are nontrivial.

To begin the illustration of this point, let us recall that *Galilei's relativity solely applies to inertial frames and establishes the equivalence of all inertial frames,* as well known.

On the contrary, *Santilli's relativities strictly apply to noninertial frames, by conception,* that is, they have been conceived to be applicable to actual physical frames of our Earthly environment which, as well known, are precisely noninertial.

Moreover, *Santilli's relativities establish equivalence subclasses of noninertial frames, those with respect to the center-of-mass frame of the system considered,* each class being characterized by each relativity (i.e., by each physical medium). The understanding is that different systems imply different subclasses of isotopically equivalent frames.

As Santilli puts it, physical events can occur in the Universe according to a multiple infinity of noninertial conditions. The isogalilean relativities essentially indicate that all these noninertial frames cannot be reduced to one single class of equivalence, but require their classification into subclasses of frames.

Moreover, Santilli has shown that only a portion of all possible noninertial frames are isotopically equivalent to the observer's noninertial frame at rest with the interior medium considered, because the remaining classes of equivalence have the more general Lie-admissible character.

But the isogalilean relativities are coverings of the conventional one. This means that the conventional inertial aspects are not lost, but fully included and actually generalized in the broader isotopic setting. In fact,

for the case of the restricted isogalilean relativities of Definition 14.2, the underlying isotransformations (14.36) are linear, thus preserving the inertial character of the reference frames.

Also, Santilli points out that such a linear–inertial character always holds for global–exterior observers, i.e., for the description of interior effects as a whole, in which case the characteristics B–functions are averaged into constants (Fig. 2 of Sect. 12). On the contrary, the nonlinear–noninertial character emerges only locally in the study of individual interior trajectories.

This elaborates in more details the abstract equivalence of Santilli's and Galilei's relativities, because the latter is indeed a global description in the indicated sense.

We now briefly outline *the application of the isogalilean relativities for the characterization of physical systems.* Recall that nonselfadjoint systems (1.1) are called "closed" when they are isolated from the rest of the Universe and therefore verify all total Galilean conservation laws.

LEMMA 14.1 (loc. cit.): Necessary and sufficient conditions for the isoinvariance of closed nonselfadjoint systems (1.1) under the symmetries $\hat{G}(3.1)$ with isounit $\hat{1}_2$ are that: 1) they can be consistently written in isospace $\hat{R}_t \times T^*\hat{E}(r,\hat{G},\hat{\mathfrak{R}})$; 2) they admit the representation in terms Birkhoff–Santilli equations

$$\frac{da^{\mu}}{dt} = \Omega^{\mu\sigma} \hat{1}_{2\sigma}{}^{\mu}(a) \frac{\partial H}{\partial a^{\nu}}, \tag{14.42}$$

where $\hat{1}_2$ is the isounit of $\hat{G}(3.1)$; and 3) the Hamiltonian is an isoscalar

$$H = p_{ia}\, \hat{G}_{ij}(t, r, p, \ldots)\, p_{ja}/2m_a + V(r_{ab}), \tag{14.43a}$$

$$r_{ab} = \{(r_{ia} - r_{ib})\, \hat{G}_{ij}(r, p, \ldots)\, (r_{ja} - r_{jb})\} \tag{14.43b}$$

Translated in a physical language, *Santilli's isogalilean relativities imply a new notion of composite system, with far reaching implications, not only classically, but also quantum mechanically.*

Recall that the conventional Galilei's relativity provides a form–invariant description of the following *closed selfadjoint systems* in $\mathfrak{R}_t \times T^*E(r,\delta,\mathfrak{R})$

$$\dot{a} = (\dot{a}^{\mu}) = \begin{pmatrix} \dot{r}_{ka} \\ \dot{p}_{ka} \end{pmatrix} = \Xi = (\Xi^{\mu}(t, a)) = \begin{pmatrix} p_{ka}/m_a \\ f^{SA}(r) \end{pmatrix} \tag{14.44a}$$

$$\dot{X}_i(t, a) = \frac{\partial X_i}{\partial a^\mu} \dot{a}^\mu + \frac{\partial X_i}{\partial t} = 0, \tag{14.44b}$$

$$k = 1, 2, 3, \qquad a = 1, 2, 3, ..., N, \quad \mu = 1, 2, ..., 6N$$

where the X's represent the familiar, total, Galilean, conserved quantities

$$X_1 = H = T(p) + V(r), \tag{14.45a}$$

$$(X_2, X_3, X_4) = (P_k) = \sum_a p_{ka}, \tag{14.45b}$$

$$(X_5, X_6, X_7) = (M_k) = \sum_a r_{ka} \wedge p_{ka}, \tag{14.45c}$$

$$(X_8, X_9, X_{10}) = (G_k) = \sum_a (m_a r_{ka} - t\, p_{ka}), \tag{14.45d}$$

The content reviewed in this volume has established that Santilli's isogalilean relativities provide instead a form-invariant description of the more general *closed nonselfadjoint systems* on $\hat{\mathfrak{R}}_t \hat{T} \times \hat{T}^* \hat{E}(\hat{r}, \hat{G}, \hat{\mathfrak{R}})$

$$\dot{a} = (\dot{a}^\mu) = \begin{pmatrix} \dot{r}_{ka} \\ \dot{p}_{ka} \end{pmatrix} \qquad \Gamma = (\Gamma^\mu(t, a, \dot{a}...))$$

$$= \begin{pmatrix} p_{ka}/m_a \\ f^{SA}{}_{ka}(r) + F^{NSA}{}_{ka}(t, r, p, \dot{p}, ...) + \int_\sigma d\sigma\, \mathfrak{F}^{NSA}{}_{ka}(t, r, p, \dot{p}, ...) \end{pmatrix} \tag{14.46a}$$

$$\dot{X}_i = \frac{\partial X_i}{\partial a^\mu} \dot{a}^\mu + \frac{\partial X_i}{\partial t} \equiv 0, \tag{14.46b}$$

$$i = 1, 2, ..., 10, \quad k = 1, 2, 3, \quad a = 1, 2, ..., N, \qquad \mu = 1, 2, ..., 6N,$$

where the X's are exactly the same as in Eq.s (14.45).

To understand this occurrence, recall that in the conventional Lie theory, the generators represent total conserved quantities. But the generators and parameters are kept unchanged as a central condition of the Lie-Santilli theory[30]. One

[30] The mathematically oriented reader can now see the far reaching physical implications of an elementary mathematical property reviewed in Sect. 3, namely, that *the basis of a*

therefore reaches the conclusion that the isotopic liftings of the conventional Galilean symmetry G(3.1) guarantee the transition from a closed selfadjoint to a closed nonselfadjoint systems.

Stated differently, generalized systems (14.46) are, in general, systems with subsidiary constraints given precisely by the total conservation laws. This is the way they were originally identified (Santilli (1978b)) and treated (Santilli (1872a)). The advance made in Santilli (1988a), (1991d) is that of removing their constrained character via the imposition of the isogalilean symmetries.

The emergence of a novel concept of composite system is then evident. It is important to indicate some of their physical implications, so as to provide the reader with an indication of the possible classical and quantum mechanical applications.

In essence, the contemporary concept of (nonrelativistic) composite system is represented, classically, by the planetary structure, and, quantum mechanically, by the atomic structure. In both cases:

a) the individual constituents are in stable orbits;

b) the interactions are at–a–distance, potential type; and

c) the center of the system can only be occupied by a mass much bigger than any of the peripheral constituents.

Santilli's composite systems are more general than the above ones. In fact,

$\hat{a}$) the individual constituents are in generally unstable orbits;[31]

$\hat{b}$) the mutual interactions are, not only at–a–distance, but also of the actual, mutual contact; and

$\hat{c}$) the center of the system, called *"Santilli's isocenter"*, can be a particle of arbitrary mass, including a mass much smaller than that of the peripheral constituents (Santilli (1988a), (1991d).[32]

The reader can now understand the physical implications. As an example, the entirety of hadron physics has been developed until now as a closed selfadjoint systems with individual constituents in stable orbits, trivially, because all these theories are conventionally Lagrangians or Hamiltonians.

But, the wavepackets of all massive particles, including the hadronic constituents, must have a dimension of at least IF, that is, of the order of the dimension of all hadrons Thus, hadrons are constituted by constituents in conditions of total mutual penetration, resulting precisely in the historical open legacy of their ultimate nonlocal structure.

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vector space is left unchanged by Santilli's isotopies.

31 Except for the case N = 2 for which only one stable orbit is admissible and that orbit is the circle. Note that the case N = 1 does not exist because one single isolated particle is free and cannot experience any interaction, whether selfadjoint or not.

32 This is only one of the many physical implications of the contact internal forces, evidently because they can literally constrain a lither particle at the center.

This renders Santilli's notion of composite system directly applicable to the hadronic structure. As a matter of fact, as clearly stated in memoir (1978a,b), he entered into the laborious process of generalizing all current relativities precisely for the purpose of attempting a more adequate representation of the hadronic structure and, possibly, the identification of the hadronic constituents with suitably altered forms of ordinary particles.

As an example, Rutherford's original conception of the neutron as a "compressed hydrogen atom" (i.e., as an ordinary electron compressed all the way to the center of the proton, say, in the core of a star) was claimed to be fundamentally inconsistent within the context of conventional quantum mechanics. On the contrary, Santilli (1989a, b, c, d) has shown that the hypothesis is fully consistent if treated via his isogalilean and isospecial relativities. In fact, Rutherford's electron cannot possibly be an ordinary Galilean or Einsteinian "center", but it can indeed be "Santilli's isocenter".

We regret the inability to review the two-body and three-body closed nonselfadjoint systems to avoid an excessive length of this volume, and we refer instead the reader to Santilli (1991d), and Jannussis, Miatovic and Veljanosky (1991).

Another effective way of appraising the possible physical relevance of any covering relativity is by identifying its implications for the characterization of a particle.

DEFINITION 14.3 (Santilli (1989)): A nonrelativistic isoparticle is an isorepresentation of one of the infinitely possible Isogalilean symmetries $\hat{G}(3.1)$ *on isospace* $\hat{\mathfrak{R}}_t \times T^*\hat{E}(r,G,\hat{\mathfrak{R}})$ *. Equivalently, a nonrelativistic isoparticle can be defined as the generalization of the conventional notion of particle induced by the isotopic liftings of the units*

$$1_t = 1 \in \hat{\mathfrak{R}}_t \Rightarrow \hat{1}_t \in \hat{\mathfrak{R}}_t, \tag{14.47a}$$

$$1 \in T^*E(r,\delta,\mathfrak{R}) \Rightarrow \hat{1}_2 = \mathrm{diag}\,(\hat{G}^{-1}, \hat{G}^{-1}) \in T^*\hat{E}(r,\hat{G},\hat{\mathfrak{R}}). \tag{14.47b}$$

$$\delta = \mathrm{diag.}\,(1,1,1) \Rightarrow G = \mathrm{diag.}\,(B_1^{-1}, B_2^{-2}, B_3^{-2}) > 0. \tag{14.47c}$$

The above definition is intended to express the need, first, to represent the actual shape of the particle considered and, consequentially, of all its infinitely possible deformations. As a result, when a particle is realistically represented, it can possess an infinite number of different intrinsic characteristics depending on the infinitely possible local conditions.

The simplest possible case is the nonrelativistic particularization of the "free relativistic isoparticle" of the preceding section, which is characterized by the the constant isometrics

$$\hat{\delta} = \hat{G} = \mathrm{diag.}\,(b_1{}^2, b_2{}^2, b_3{}^2), \qquad b_i = \text{constants} > 0. \tag{14.3}$$

and Hamilton-Santilli equations

$$\dot{r}_i = b_i^{-2}\,\partial H / \partial p_i = p_i/m = v_i, \tag{14.4a}$$

$$\dot{p}_i = -\,b_i^{-2}\,\partial H / \partial r_i = 0. \tag{14.4b}$$

Thus, *the isogalilean equations of motion are, in this case, identical to those of the conventional Galilei's relativity.* Nevertheless, the use of the isogalilean relativities permits the direct representation of:

1) the extended character of the particle;
2) the actual shape of the particle considered; and
3) an infinite class of possible deformations of the original shape;

all the above already at this primitive, classical, nonrelativistic level33.

An endless number of additional examples can then be worked out with the inclusion of any combination of conventional, local-potential, as well as nonlocal-nonpotential interactions (see the exampleS at the enD of Sect. 7, and Santilli (1991d).

By recalling the "Theorems of Direct Universality" of Santilli's isotopic formulations for systems (1.1), we have the following important property:

PROPOSITION 14.1: Whenever applicable, all Santilli's general and restricted isogalilean relativities are exactly valid for the systems of our physical reality.

To state it differently,

Santilli's classical isogalilean relativities do not need any experimental verification because verified by construction by the systems of our physical reality.

We can now reverse our order of presentation as follows:

I) assume the isogalilean relativities as the fundamental ones;
II) construct the isospecial relativities as derived from a first generalization;

and

33 In addition, there is the emergence of a form of mass-renormalization also within a purely classical context (see Santilli (1991d), Appendix III.A).

III) construct the isogeneral relativities as the final generalization on isocurved isospaces.
with evident enclosure properties

SANTILLI'S		***SANTILLI'S***		***SANTILLI'S***
ISOGENERAL RELATIVITIES	$\supset$	*ISOSPECIAL* ***RELATIVITIES***	$\supset$	***ISOGALILEAN RELATIVITIES***

Then,

Owing to the deep compatibility and inter-relations of Santilli's covering relativities, the established exact validity of the isogalilean relativities provide rather solid grounds for the exact validity of the remaining isospecial and isogeneral relativities.

The understanding is that the experiments proposed in the literature for the verification of the isospecial relativities are necessary for the final resolution of the issue.

I.15: ISOROTATIONAL SYMMETRIES

In this section we shall review Santilli's infinite family of classical, isotopic generalizations $\hat{O}(3)$ of the rotational symmetry O(3) on isoeuclidean spaces $\hat{E}(r,\hat{\delta},\hat{\mathfrak{R}})$, called *Santilli's isorotational symmetries*, or *rotational-isotopic symmetries*.

Isosymmetries $\hat{O}(3)$ were introduced, apparently for the first time, in Santilli (1978a); expanded in Santilli (1982a)l and finally studied in details in Santilli (1985b) in their abstract, and therefore nonlinear and nonlocal version. The classical nonlinear and nonlocal realizations of $\hat{O}(3)$ were studied for the first time in Santilli (1988a) and then expanded in Santilli (1991d).

Regrettably, we shall be unable to present, for brevity, the isorepresentation theory of $\hat{O}(3)$, which has been studied within the context of the covering isounitary symmetries $\hat{SU}(2)$ in Santilli ((1989), jointly with other liftings of the conventional theory, e.g., the *iso-Clebsh-Gordon coefficients*, etc.

We believe that this topic is the final necessary complement of the analysis of this volume, because it constitutes the central part of Santilli's isogeneral, isospecial and isogalilean relativities.

Stated in a nutshell, *the conventional rotational symmetry provides* a

theory of rigid bodies, as well known. *Santilli's covering isorotational theory provides instead a theory for extended and therefore deformable bodies.* In particular, *the theory of isorotations establishes the preservation of the exact O(3) symmetry for all infinitely possible deformations of the sphere*, as clearly established in Santilli (1985b), contrary to a rather widespread, erroneous belief in both mathematical and physical circles.

We felt obliged to mention in the preceding sections the scarcity of independent investigations on Santilli's isotopies. We should now stress that *Santilli is the originator (in 1978) and sole contributor in both the $\hat{O}(3)$ and $\hat{O}(2)$ symmetries to this writing (1992), without any contribution by independent researchers*.

At first, this author could not believe such an occurrence, because the rotational symmetry is the most fundamental part of all of contemporary physics. The birth of a structural generalization of the rotational symmetry with fundamentally novel capabilities was not expected to remain ignored. But that's as it has been. In fact, library searches and consultations with experts in the field have confirmed the lack of independent contributions on isosymmetries $\hat{O}(3)$ and $\hat{O}(2)$ to this writing.

This review has been derived from Sect. III.3 of Santilli (1991d). To begin, it appears recommendable to outline first the main results of the abstract formulation of isorotations, and then pass to their classical realization. A necessary prerequisite for the understanding of this section is a knowledge of Sect.s 1 to 9.

DEFINITION 15.1 (Santilli (1985a), (1988a)): The "rotational isotopic groups" $\hat{O}(3)$, or "isorotational groups", are the largest possible isolinear and isolocal groups of isometries of the isoeuclidean spaces

$$\hat{E}(r,\hat{\delta},\hat{\mathfrak{R}}): \ \hat{\delta} = T(r, \dot{r}, \ddot{r}, \ldots)\, r, \ \hat{\mathfrak{R}} = \mathfrak{R}\hat{1}, \ \hat{1} = T^{-1} = \hat{\delta}^{-1} \tag{15.1a}$$

$$\det. T = \det. \hat{\delta} \neq 0, \quad T = T^{\dagger}, \tag{15.1b}$$

$$(r \,\hat{,}\, r) = (r,\hat{\delta}r)\hat{1} = (\hat{\delta}r,r)\hat{1} = \hat{1}(r,\hat{\delta}r) = [r^i \hat{\delta}_{ij}(r, \dot{r}, \ddot{r}, \ldots)\, r^j]\hat{1}, \tag{15.1c}$$

characterized by: the right, modular-isotopic transformations

$$r' = \hat{R}(\theta) * r = \hat{R}(\theta)\,\hat{\delta}\, r, \quad \hat{\delta} = \text{fixed}, \tag{15.2}$$

where the θ's are the conventional Euler's angles, whose elements $\hat{R}(\theta)$ verify the properties

$$\hat{R} * \hat{R}^t = \hat{R}^t * \hat{R} = \hat{1}, \tag{15.3}$$

or, equivalently, $\hat{R}^t = \hat{R}^{-1}$, *and verify the isotopic group rules*

$$\hat{R}(0) = \hat{1} = \hat{\delta}^{-1}, \tag{15.4a}$$

$$\hat{R}(\theta) * \hat{R}(\theta') = \hat{R}(\theta') * \hat{R}(\theta) = \hat{R}(\theta + \theta'), \tag{15.4b}$$

$$\hat{R}(\theta) * \hat{R}(-\theta) = \hat{1}, \tag{15.4c}$$

Equivalently, the isorotational groups $\hat{O}(3)$ *can be defined as the isosymmetries of the infinitely possible deformations of the sphere representable via the particular realization of the isometric*

$$\hat{\delta} = \mathrm{diag.}\,(g_{11}, g_{22}, g_{33}), \tag{15.5a}$$

$$r^2 = r_1\, g_{11}\, r_1 + r_2\, g_{22}\, r_2 + r_3\, g_{33}\, r_3\,, \tag{15.5b}$$

Isogroups $\hat{O}(3)$ resulted to be tridimensional simple Lie groups which can be constructed from the sole knowledge of the isometric $\hat{\delta}$ via the conventional (matrix) generators and parameters of the rotational group O(3).

From Eq. (15.3) it is easy to see that isorotations satisfy the conditions

$$\det(\hat{R}\hat{\delta}) = \pm 1. \tag{15.6}$$

Therefore, $\hat{O}(3)$ is characterized by a continuous semisimple subgroup denoted $S\hat{O}(3)$ for the case $\det(\hat{R}\hat{\delta}) = +1$, and a discrete invariant part for the case $\det(\hat{R}\hat{\delta}) = -1$ representing the isoinversions. (see below).

Each one of the infinitely many possible $S\hat{O}(3)$ subgroups can be essentially characterized as follows. The abstract, enveloping isoassociative algebra $\hat{\xi}$ of Sect. 6 is now realized in the form $\hat{\xi}$ characterized by the isounit $\hat{1}$, the conventional generators J_k, k = 1, 2, 3, of SO(3) in their fundamental, 3×3 representation, and all their possible polynomials, resulting in the infinite dimensional basis of the Poincaré-Birkhoff-Santilli-Witt Theorem

$$\hat{\xi}(\hat{\mathbf{SO}}(3)): \quad \hat{1}, \quad J_k, \quad J_i * J_j \ (i \leq j), \quad J_i * J_j * J_k \ (i \leq j \leq k), \ \ldots\ldots\ldots\ldots \tag{15.7}$$

The isocommutation rules of the Lie-isotopic algebra $\hat{\mathbf{SO}}(3)$ of $\hat{SO}(3)$ were also studied (*loc. cit.*) and shown to be reducible to the form

$$[\mathfrak{J}_i \hat{,} \mathfrak{J}_j] = \mathfrak{J}_i * \mathfrak{J}_j - \mathfrak{J}_j * \mathfrak{J}_i = \mathfrak{J}_i \hat{\delta} \mathfrak{J}_j - \mathfrak{J}_j \hat{\delta} \mathfrak{J}_i = \epsilon_{ijk} \mathfrak{J}_k, \tag{15.8}$$

under a suitable redefinition $\mathfrak{J}_k$ of the generators J_k (see below), where the tensor ϵ_{ijk} is the conventional totally antisymmetric tensor characterizing the structure constants of $\mathbf{SO}(3)$.

The Lie–Santilli groups $\hat{SO}(3)$ were obtained via an isoexponentiation of structure (15.7) in $\hat{\xi}$, resulting in the expression

$$\hat{SO}(3): \quad \hat{R}(\theta) = \{e_{|\hat{\xi}}^{J_1\theta_1}\} * \{e_{|\hat{\xi}}^{J_2\theta_2}\} * \{e_{|\hat{\xi}}^{J_3\theta_3}\}, \tag{15.9}$$

which can be rewritten in the conventional associative envelope ξ of $\mathbf{SO}(3)$

$$\hat{SO}(3): \quad \hat{R}(\theta) = (\prod_{k=1,2,3} e_{|\xi}^{J_k \hat{\delta} \theta_k}) \hat{1} = \hat{1} (\prod_{k=1,2,3} e_{|\xi}^{\theta_k \hat{\delta} J_k})$$

$$\stackrel{\text{def}}{=} [S_{\hat{\delta}}(\theta)]\hat{1} = \hat{1}[S_{\hat{\delta}}{}^{t}(\theta)]. \tag{15.10}$$

The isorotations can then be written in the simpler form

$$r' = \hat{R}(\theta) * r \equiv S_{\hat{\delta}}(\theta)\, r, \tag{15.11}$$

which is useful for computational convenience. The understanding is that the mathematically correct form remains the isotopic form (to prevent the violation of the linearity condition).

The discrete part is characterized by the now familiar *isoinversions*

$$\hat{P} * r = P\, r = -\, r, \tag{15.12}$$

where P characterizes the conventional discrete components of O(3).

The notion of isorotation groups was turned into that of *isorotational*

symmetries by noting that, under the conditions of Definition 15.1, isotransformations (15.4) leave invariant, by construction, the separation in $\hat{E}(r,\hat{\delta},\hat{\mathfrak{R}})$ (Theorem 8.1), i.e.,

$$r'^{\hat{2}} = (r'^{i}\,\hat{\delta}_{ij}\,r'^{j})\,\hat{1} \equiv (r^{i}\,\hat{\delta}_{1ij}\,r^{j})\,\hat{1} = r^{\hat{2}} \tag{15.13}$$

The above result holds in view of the property

$$S_{\hat{\delta}}^{\,t}\,\hat{\delta}\,S_{\hat{\delta}} \equiv \hat{\delta}, \tag{15.14}$$

which is identically verified for all possible metrics $\hat{\delta}$ of the class admitted, plus similar identities for the isoinversions.

The capability for the isorotational symmetries $\hat{O}(3)$ to leave invariant all possible ellipsoidical deformations of the sphere, Eq.s (15.5), then trivially follows from invariance (15.13).

By using Eq.s (15.9) or (3.10), it is easy to compute a general isorotation (8.66) around the third axis, i.e.,

$$\hat{\delta} = \text{diag.}\,(g_{11}, g_{22}, g_{33}), \tag{15.15a}$$

$$S_{\hat{\delta}}(\theta) = \begin{pmatrix} \cos[\theta_3(g_{11}g_{22})^{\frac{1}{2}}] & g_{22}(g_{11}g_{22})^{\frac{1}{2}}\sin[q_3(g_{11}g_{22})^{\frac{1}{2}}] & 0 \\ -g_{11}(g_{11}g_{22})^{\frac{1}{2}}\sin[\theta_3(g_{11}g_{22})^{\frac{1}{2}}] & \cos[\theta_3(g_{11}g_{22})^{\frac{1}{2}}] & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The above notion of abstract isorotational symmetry then leads to the following property anticipated in Sect. 8:

LEMMA 15.1 (Santilli (loc. cit.): The abstract isotope $\hat{O}(3)$ of $O(3)$ with a *nowhere singular, Hermitean and diagonal isometric (15.5a) of unspecified signature provides* a *single geometric unification of all possible simple, two-dimensional, Lie groups of Cartan's classification.*

All physical applications of the isorotational symmetries, and their use within the context of isotopic relativities in particular, were restricted by Santilli to the infinite class of isotopes

$$\hat{O}(3): \quad \text{sig.}\,\hat{\delta} = (+1, +1, +1), \quad \hat{\delta} > 0. \tag{15.16}$$

This essentially implies the restriction only to those isotopes $\hat{O}$ that are locally isomorphic to O(3). Only the isoduals

$$\hat{O}^d(3): \qquad \text{sig.}\,\hat{\delta} = (-1, -1, -1), \quad \hat{\delta} < 0, \tag{15.17}$$

were considered by Santilli for the construction of the corresponding isodual relativities which we have not considered for brevity[34].

The terms "isorotations" or "rotational-isotopic transformations" are therefore referred to in this volume, specifically, to those characterized by positive-definite isometrics $\hat{\delta}$.

By recalling that all nonsingular and Hermitean metrics and isometrics can be diagonalized, all positive-definite isometrics can therefore be written in the diagonal form

$$\hat{\delta} = \text{diag.}\,(b_1{}^2, b_2{}^2, b_3{}^2), \tag{15.18a}$$

$$b_k = b_k(t, r, p, \dot{p}, \ldots) > 0, \tag{15.18b}$$

which is assumed hereon as the basic form.

The first physical motivation for the restrictions of the isometrics $\hat{\delta}$ to be positive-definite is the following. As well known, *mathematically* we can indeed deform the sphere

$$r^2 = r_1 r_1 + r_2 r_2 + r_3 r_3 > 0, \tag{15.19}$$

into all infinitely possible compact (ellipsoidical) and noncompact (hyperboloid) forms

$$r^{\hat{2}} = r_i\, g_{ii}\, r_i \gtreqless 0, \tag{15.20}$$

[34] Santilli's notion of isoduality has truly intriguing possibilities in particle physics because it permits the possibility of reconstructing exact discrete symmetries, of course, at the isotopic level, when believed to be conventionally broken. This includes the possibility of reconstructing parity as exact symmetry for weak interactions via the simple embedding of all parity-violating terms in the isounit as well as in the additional degrees of freedom of operator formulations offered by the isotopy of the Hilbert space (Santilli (1983b), (1985c), (1989). Note the necessary use of the Lie-Santilli theory for the very formulation of the notion of isoduality, let alone its treatment, because it requires the nontrivial isotopic product ATB − BTA = A (−I) B − B(−I) A.

which then produce the classification of all possible, compact and noncompact isotopes reviewed in Sect. 6.

However, on physical grounds, a given sphere can only be deformed into ellipsoids, and there exists no known physical process capable of turning a sphere into a hyperboloid.

Additional reasons are of geometrical nature, and are motivated by Santilli intent of reaching the unification of isotopic and conventional theories at the abstract, realization-free level.

Along these lines, one note that a most salient geometric axiom of the conventional theory of O(3) is the positive-definiteness of its invariant, Eq.s (15.19). In order to achieve an isotopic theory of $\hat{O}(3)$ capable of coinciding with that of O(3) at the abstract level, Santilli therefore preserves the same axiom.

Some of the main properties of isorotations can then be expressed as follows

THEOREM 15..1 (Loc. cit.): Santilli's isorotational symmetries $\hat{O}(3)$ of all infinitely possible ellipsoidical deformations of the sphere on the isoeuclidean spaces $\hat{E}(r,\hat{\delta},\hat{\mathfrak{R}})$, $\hat{\mathfrak{R}} = \mathfrak{R}\hat{1}$, $\hat{1} = \hat{\delta}^{-1}$, $\hat{\delta} > 0$, *verify the following properties:*

1) The groups $\hat{O}(3)$ consist of infinitely many different groups corresponding to the infinitely many possible deformations of the sphere (explicit forms of the isometric $\hat{\delta}$), Eq. (3.18a);

2) All isosymmetries $\hat{O}(3)$ are locally isomorphic to O(3) under conditions (3.18b) herein assumed; and

3) The groups $\hat{O}(3)$ constitute "isotopic coverings" of O(3) in the sense that:

3.a) The formers are constructed via methods (the Lie-isotopic theory) structurally more general than those of the latter (the conventional Lie's theory);

3.b) The formers represent physical conditions (deformations of the sphere; inhomogeneous and anisotropic interior physical media; etc.) which are broader than those of the conventional symmetry (perfectly rigid sphere; homogeneous and isotropic space; etc.); and

3.c) All groups $\hat{O}(3)$ recover O(3) identically whenever $\hat{1} = I$ and they can approximate the latter as close as desired for $\hat{1} \approx I$.

A first illustration of the nontriviality of the above results can be expressed via the property indicated at the beginning of this sections, namely, that the isorotational symmetries reconstruct as exact the O(3) symmetry for all possible deformations of the sphere.

Note that, for general ellipsoids (15.18) *the "rotational symmetry" is exact, but the "conventional rotations" do not constitute a symmetry any longer.* This occurrence is at the foundation of the need for Santilli's isorelativities.

The latter aspect is rendered necessary by the following property.

COROLLARY 15.1.1 (loc. cit.): While conventional rotations are trivially linear and local in $E(r,\delta,\mathfrak{R})$, Santilli's isorotations are formally isolinear and isolocal in $\hat{E}(r,\hat{\delta},\hat{\mathfrak{R}})$, but generally nonlinear and nonlocal in $E(r,\delta,\mathfrak{R})$, i.e.,

$$r' = \hat{R}(\theta)*r = \hat{R}(\theta)\,\hat{\delta}(t, r, \dot{r}, \ddot{r},...)\,r \tag{15.21}$$

A further important result is the isotopic generalization of the conventional *Euler's theorem* on the general displacement of a rigid body with one point fixed (see, e.g., Goldstein (1950)), which we can express via the following

THEOREM 15.2 (loc. cit.): The general displacement of an elastic body with one fixed point is an isorotation $\hat{O}(3)$ around an axis through the fixed point.

In different terms, isorotations characterize not only a rotation of a given body, but also, jointly, its possible deformations. Thus, while the theory of isorotations characterizes *elastic bodies*, as indicated earlier.

This completes our review of the abstract treatment of the isorotational symmetry $\hat{O}(3)$. We are now sufficiently equipped to review Santilli's classical realizations of the isorotations $\hat{O}(3)$ under the conditions of: 1) being directly applicable to classical, closed, nonrelativistic, nonselfadjoint systems (14.46); 2) permitting the achievement of the conservation of the total angular momentum via the invariance of the systems under isorotations (without any need of subsidiary constraints); and 3) allowing the inclusion of nonlocal internal forces.

The phase space of the theory is the cotangent bundle $T^*\hat{E}(r,\hat{\delta},\hat{\mathfrak{R}})$ with the familiar local coordinates

$$a = (a^{\mu}) = (r_{ka}, p_{ka}), \tag{15.22}$$

$$\mu = 1, 2, ..., 6N,\ k = 1, 2, 3\ (= x, y, z),\ a = 1, 2, ..., N$$

Isospaces $T^*\hat{E}(r,\hat{\delta},\hat{\mathfrak{R}})$ are then equipped with the one–isoforms (Sect. 9)

$$\Phi_1 = R^\circ_\mu \,\hat{T}_1{}^\mu{}_\nu \, da^\nu = p_{ia}\, \delta_{ij}\, dr_{ja}, \tag{15.23a}$$

$$R^\circ = (p, 0), \qquad \hat{T}_1 = (\hat{T}_1{}^\mu{}_\nu) = \text{diag.}\,(\hat{\delta}, \hat{\delta}), \tag{15.23b}$$

which is the fundamental space for the representation of systems (14.66) via Pfaffian-Santilli variational principles.

To study the isosymmetries of the systems we have to consider the isospace $T^*\hat{E}_2(r,\hat{\delta},\hat{\Re})$ of two-isoforms constructed from one-isoforms

$$\Phi_2 = \partial\Phi_1 = \tfrac{1}{2}\,\omega_{\mu\alpha}\,\hat{T}_2{}^\alpha{}_\nu\, \partial a^\mu \hat{\wedge} \partial a^\nu , \tag{15.24}$$

where $\omega_{\mu\nu}$ is the canonical symplectic tensor, and the isotopic element $\hat{T}_2$ is generally different than $\hat{T}_1$, with the explicit form (9.94)

$$(\hat{T}_2{}^\mu{}_\nu) = [\, b^{2\,\mu}\, b^2{}_\nu + \omega^{\mu\rho}\,(R^\circ{}_\nu \frac{\partial b^2{}_\nu}{\partial a^\rho} - R^\circ{}_\rho \frac{\partial b^2{}_\rho}{\partial a^\nu})\,] =$$

$$\stackrel{\text{def}}{=} \text{diag}\,(\hat{G}, \hat{G}), \quad \hat{G} = \text{diag.}\,(B_1{}^2, B_1{}^2, B_3{}^2) > 0, \tag{15.25a}$$

$$(b^2{}_\mu) = (b^{2\mu}) = (\hat{\delta}, \hat{\delta}) \tag{15.25b}$$

The isospace used in the classical isosymmetries is then given by

$$T^*\hat{E}_2(r,\hat{\delta},\hat{\Re}) \equiv T^*\hat{E}(r,\hat{G},\hat{\Re}) : \; \hat{r} = [r_{ia}\hat{G}_{ij}(t, r, p, \dot{p}, \dots)\, r_{ja}\,]\hat{1}_2, \tag{15.26a}$$

$$\hat{1}_2 = (\hat{T}_2)^{-1} = \text{diag.}\,\hat{G}^{-1}, \hat{G}^{-1}), \tag{15.26b}$$

Thus, the actual invariant of the isorotational theory under study is invariant (15.26a).

By recalling the interplay between geometry and algebras of Sect. 9, the Lie-Santilli brackets of the theory are given by

$$[A \,\hat{,}\, B] = \frac{\partial A}{\partial a^\mu}\, \omega^{\mu\alpha}\, \hat{1}_{2\alpha}{}^\nu\, \frac{\partial B}{\partial a^\nu} \tag{15.27}$$

$$= \frac{\partial A}{\partial r_{ka}} B_k^{-2}(t, r, p, ..) \frac{\partial B}{\partial p_{ka}} - \frac{\partial B}{\partial r_{ka}} B_k^{-2}(t, r, p, ..) \frac{\partial A}{\partial p_{ka}} .$$

Our objective is that of reviewing the theory of isorotations $\hat{O}(3)$ via brackets (15.27). For clarity, we shall proceed in stages, and begin with the study first of the case of constant isometrics

$$\hat{\delta} = \text{diag.}(b_1^2, b_2^2, b_3^2), \tag{15.28a}$$

$$b_k = \text{constants} > 0, \tag{15.28b}$$

for which $\hat{T}_1 \equiv \hat{T}_2$, and the Lie-isotopic brackets (15.27) assume the simpler form

$$[A \hat{,} B] = \frac{\partial A}{\partial r_{ka}} b_k^{-2} \frac{\partial B}{\partial p_{ka}} - \frac{\partial B}{\partial r_{ka}} b_k^{-2} \frac{\partial A}{\partial p_{ka}}, \tag{15.29}$$

To identify the Lie-Santilli algebra $\hat{SO}(3)$ characterized by brackets (15.29), Santilli computes first the *fundamental Birkhoffian commutation rules* which are readily given by

$$([a^\mu \hat{,} a^\nu]) = \begin{pmatrix} [r_i \hat{,} r_j] & [r_i \hat{,} p_j] \\ [p_i \hat{,} r_j] & [p_i \hat{,} p_j] \end{pmatrix} = (\hat{\Omega}^{\mu\nu}) = \begin{pmatrix} 0 & \hat{\delta}^{-1} \\ -\hat{\delta}^{-1} & 0 \end{pmatrix} \tag{15.30}$$

Next, Santilli introduces the generators of the Lie-isotopic algebra $\hat{SO}(3)$ of $\hat{SO}(3)$ which, as now familiar, are given by the *conventional generators* of $O(3)$, i.e., by the components of the angular momentum

$$J_k = \epsilon_{kij}\, r_i\, p_j, \tag{15.31}$$

Santilli calls the above quantities the components of the *Birkhoffian angular momentum* to emphasize the fact that they characterize a generalized notion because no longer defined on $T^*E_2(r,\delta,\Re)$, but on $T^*\hat{E}_2(r,\hat{\delta},\hat{\Re})$.

Thus, while the magnitude of the Hamiltonian angular momentum is given by the familiar expression

$$J^2 = J_k J_k, \tag{15.32}$$

the magnitude of the Birkhoffian angular momentum is instead given by

$$J^{\hat{2}} = J{*}J = J_i \hat{\delta}_{ij} J_j = J_k b_k^{\ 2} J_k. \tag{15.33}$$

Note that the interpretation of components (15.31) as isoscalars in $\hat{\mathfrak{R}}$ would imply the expressions

$$\mathfrak{J}_k = \hat{\epsilon}_{kij} * \hat{r}_i * \hat{p}_j = (\epsilon_{kij} r_i p_j)\hat{1} = J_k \hat{1}, \tag{15.34}$$

called the *trivial isotopy* (Sect. 8) because it does not provide a generalized invariance, as the reader is encouraged to verify.

Also, the reader should keep in mind that we are dealing with the classical realization of $\hat{SO}(3)$, rather than its matrix realization as in Santilli (1985b). This implies that the generators of the isosymmetries must be ordinary functions, while quantities (15.34) are matrices.

To compute the isocommutation rules of $\hat{SO}(3)$, Santilli first computes the isocommutation rules between the angular momentum, and the local variables, resulting in the expressions

$$[J_k \hat{,} r_i] = \epsilon_{kij} b_i^{-2} r_j, \tag{15.35a}$$

$$[J_k \hat{,} p_i] = \epsilon_{kij} b_i^{-2} p_j, \tag{15.35b}$$

(where there is evidently no summation on the i-index).

The desired isocommutation rules of the (compact) isorotational algebra $\hat{SO}(3)$ are then given by (Santilli (198*a), (1991d))

$$\hat{SO}(3): \quad [J_i \hat{,} J_j] = C_{ij}^{\ k} J_k = \epsilon_{ijk} b_k^{-2} J_k. \tag{15.36}$$

which, under the redefinition

$$\mathfrak{J}_1 = b_2 b_3 J_1, \quad \mathfrak{J}_2 = b_1 b_3 J_2, \quad \mathfrak{J}_3 = b_1 b_2 J_3, \tag{15.37}$$

can be written

$$[\hat{J}_i \hat{,} \hat{J}_j] = \epsilon_{ijk} \hat{J}_k, \tag{15.38}$$

This confirms the existence of a classical realization of the isocommutation rules of $\hat{SO}(3)$ possessing the same structure constants of SO(3). In turn, this confirms the local isomorphism between all possible isotopes $\hat{SO}(3)$ of class (15.28) and SO(3) in accordance with Theorem 3.1.

The isocenter of the enveloping algebra (Sect. 6) is given by the isounit, which is the zero-order isocasimir, $C^{(o)} = \hat{1}$, and magnitude (15.33) of the Birkhoffian angular momentum, $C^{(2)} = J^{\hat{2}}$, as expected. In fact,

$$[J^{\hat{2}} \hat{,} J_i] = [J_k b_k^2 J_k \hat{,} J_i] = 2\epsilon_{kij} J_k J_j \equiv 0. \tag{15.39}$$

Note that the isosquare of J has the particular geometrical significance

$$J^{\hat{2}} = (\det \hat{\delta})\, J^2, \tag{15.40}$$

with intriguing implications in particle physics we hope to indicate at some future time[35].

Note also that $J^2 = J_k J_k$ *is not* an isocasimir of Lie–Santilli algebra (15.36) or of (15.38), as the reader can verify. This occurrence is important inasmuch as it confirms the correctness of isosquare (15.33).

The occurrence also indicates that expression (15.38) of the isocommutation rules has a primary *mathematical* significance, inasmuch as it is formally identical to the conventional commutation rules. However, the isocommutation rules of direct *physical* significance are those in the physical angular moments J, i.e., rules (15.36).

The classical realization of the Lie-Santilli group $\hat{SO}(3)$ by

$$\hat{SO}(3): \quad \hat{R}(\theta) = [\, \prod_{k=1,2,3} e_{|\xi}{}^{\theta_k \omega^{\mu\alpha} \hat{1}_{2\alpha}{}^{\nu} (\partial J_k/\partial A^{\nu})(\partial/a^{\mu})} \,]\, \hat{1}$$

[35] It essentially implies a possible mutation of the conventional discrete values of the spin 0, ± ½, ± 1, ..., into mutated values 0, ± ½ $\hat{\Delta}$, ± 1 $\hat{\Delta}$, ..., $\hat{\Delta} = \det \hat{\delta}$, which is however valid only for *ONE ISOPARTICLE INSIDE HADRONIC MATTER*, such as a proton in the core of a star undergoing gravitational, collapse, and not for an isolated particle in vacuum (Santilli (1989b)). This possible internal mutation of spin is then at the basis of the interpretation of the total spin ½ for Rutherford's historical conception of the neutron as a "compressed hydrogen atom", as well as for the possible interpretation of the hadronic constituents with physical, ordinary particles freely produced in spontaneous decays (Santilli (1978b), (1981a)).

$$\stackrel{\text{def}}{=} S_{\hat\delta}(\theta)\hat 1, \tag{15.41}$$

where the exponentials are expanded in the conventional associative envelope ξ for simplicity.

Note the true realization of the notion of isotopic lifting of a Lie symmetry, i.e., the preservation of the original generators and parameters of the symmetry, and the isotopic generalization of the *structure* of the Lie group via the liftings $1 \Rightarrow \hat 1_2$.

The computation of examples is straightforward. For instance a (classical) *isorotation around the third axis* is given by (Santilli (*loc. cit.*))

$$r' = \hat R(\theta_3)\ast r = S_{\hat\delta}(\theta_3)\, r = \tag{15.42}$$

$$\begin{pmatrix} r'_1 \\ r'_2 \\ r'_3 \end{pmatrix} = \begin{pmatrix} r_1\cos(\theta_3 b_1 b_2) - r_2 \dfrac{b_2}{b_1}\sin(\theta_3 b_1 b_2) \\ r_1 \dfrac{b_1}{b_2}\sin(\theta_3 b_1 b_2) + r_2\cos(\theta_3 b_1 b_2) \\ r_3 \end{pmatrix}$$

The proof of the invariance of isoseparation (15.28) under the above transformation is an instructive exercise for the reader interested in acquiring a knowledge of Santilli's relativities. The computation of other examples can be readily done via Eq.s (15.41).

Note that the convergence of series (15.41) into finite transformations of type (15.42) is reduced to the convergence of the original series prior to the lifting.

Note also the appearance of the isotopic elements b_k directly in the angles of isorotation. This occurrence is useful for the reconstruction of the exact rotational symmetry according to the rule

$$\theta_3|_{\text{Ham.}} = \theta_3|_{\text{Birk.}}\, b_1 b_2, \tag{15.43}$$

which has important applications in particle physics (see the Figure 5 below)

In different terms, *the deformation experienced by the body considered, and represented by the b-quantities, is compensated by the isorotation in such a way that the combination of the deformation and isorotation equals the angle of*

rigid rotation. In this way, the exact rotational symmetry of a rigid body, the l.h.s. of Eq.s (15.49), is decomposed into a product of an isorotation and the b-quantities of the same value.

This is the realization in Birkhoff-Santilli mechanics of the property that all distinctions between conventional and isotopic symmetries cease to exist at the abstract, realization-free level.

We now pass to the application of the general theory of *isoinvariance* outlined in Sect. 8, to the isorotation of closed nonselfadjoint systems (14.66). For this purpose, we have to verify first that the J's are indeed the generators of isorotations.

Consider an infinitesimal isorotation $\delta\theta$ around a fixed axis with unit isovector $n = (n_1, n_2, n_3)$ in $\hat{E}(r,\hat{\delta},\hat{\Re})$, i.e.,

$$r_k \Rightarrow r'_k + \delta\theta\, \epsilon_{kij}\, n_i\, r_j, \tag{15.44a}$$

$$p_k \Rightarrow p_k + \delta\theta\, \epsilon_{kij}\, n_i\, p_j. \tag{15.44b}$$

The isoexponentiation of the above quantities yields the relations

$$\{e_{|\xi}^{-\delta\theta\, n*J}\}\, r_k \approx r_k - \delta\theta\, [n*J \hat{,} r_k] = r_k + \delta\theta\, \epsilon_{kij}\, n_i\, r_j, \tag{15.45a}$$

$$\{e_{|\xi}^{-\delta\theta\, n*J}\}\, p_k \approx p_k - \delta\theta\, [n*J, p_k] = p_k + \delta\theta\, \epsilon_{kij}\, n_i\, p_j, \tag{15.45b}$$

where the $*$ product is evidently that in $\hat{E}(r,\hat{\delta},\hat{\Re})$. This confirms that the conventional components of the angular momentum are indeed the generators of the isorotations.

The notion of *isorotational symmetry* is then given by a simple isotopy of the conventional one (Definition 8.3). In fact a Birkhoffian $B(r, p)$ is invariant under an isorotation around the n-axis iff it verifies the invariance property

$$\begin{aligned} B(r, p) &= B(r + \delta\theta\, n\hat{\wedge}J,\ p + \delta\theta\, n\hat{\wedge}J) \\ &= \{ e_{|\xi}^{-\delta\theta\, n\hat{\wedge}J} \}\, B(r, p), \end{aligned} \tag{15.46}$$

where $\hat{\wedge}$ is the vector product computed in $T^*\hat{E}(r,\hat{\delta},\hat{\Re})$, which can hold iff

$$[J_k \hat{,} B] = 0, \quad k = 1, 2, 3, \tag{15.47}$$

For a more rigorous and general presentation, see Theorems 8.2 and 8.3.

We reach in this way the rather simple conclusion that a *Birkhoff–Santilli vector-field is invariant under isorotations when properly written in $T^*\hat{E}(r,\hat{\delta},\hat{\mathfrak{R}})$*, i.e., when all operations of contraction, power, etc. are properly made with the isometric $\hat{\delta}$, as in the following Hamiltonians

$$B = H = T(p) + V(r) = \frac{p_{ia}\,\hat{\delta}_{ij}\,p_{ja}}{2m_a} + V(r), \tag{15.48a}$$

$$r = |r_{ia}\,\hat{\delta}_{ij} r_{ja}|^{\frac{1}{2}}. \tag{15.48b}$$

Finally, note from Theorem 8.3, that *conditions (15.47) are necessary and sufficient for the complete invariance of nonlinear, nonlocal, nonhamiltonian and nonnewtonian systems (14.66) represented via the Hamilton–Santilli equations.*

We now pass to a review of isometrics with a nontrivial functional dependence, namely, for general brackets (15.27).

It is easy to see that the isocommutation rules remain structurally unchanged under the generalization herein considered, with the only replacement of the b– with the B–quantities, e.g.,

$$[J_i \hat{,} r_j] = \epsilon_{ijk}\, B_j^{-2}\, r_k, \quad [J_i \hat{,} p_j] = \epsilon_{ijk}\, B_j^{-2}\, p_k. \tag{15.49}$$

The *general isocommutation rules* of *Santilli's isorotational algebras* $\hat{SO}(3)$ are then given by

$$\hat{SO}(3): \quad [J_i \hat{,} J_j] = C_{ij}^{\ k}(r, p, \ldots)\, J_k = \epsilon_{ijk}\, B_k^{-2}(r, p, \ldots)\, J_k, \tag{15.50}$$

and provides another illustration of the *structure functions* of the Lie–Santilli theory (Sect. 6). The reformulation of the above algebra to reach the same structure constants of the conventional symmetry, as in Eq.s (15.38), is here left as an instructive exercise for the interested reader.

As one can see, under the condition of positive–definiteness of the isometric G, all infinitely possible isotopes $\hat{SO}(3)$ remain isomorphic to SO(3), by therefore preserving the semisimple and connected properties of SO(3).

The study of the *global isocasimir invariants*, that is, the isocasimirs valid everywhere in $T^*\hat{E}_2(r,\hat{\delta},\hat{\mathfrak{R}})$, under a nontrivial functional dependence of the isometric, is involved on technical grounds, inasmuch as it requires a deeper knowledge of the Birkhoffian realization of universal enveloping isoassociative algebras and related isoneutral elements (see the remarks at the end of Sect. 6)..

It is easy to see that, in this local sense, the isocasimirs of realization (15.50) persist, i.e., are given by

$$C^{(0)} = \hat{1}_2\big|_{\bar{a}}, \qquad C^{(2)} = \hat{J}^2 = (J\,\hat{G}\,J)\big|_{\bar{a}} \tag{15.51}$$

A simple example of a global isocasimir is given when

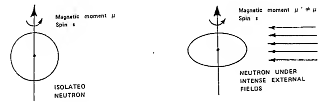
$$B_1 = B_2 = B_3 \equiv B(p), \quad \hat{T}_2 = B^2 I, \quad \hat{1}_2 = B^{-2}\, I, \tag{15.52}$$

in which case the magnitude of the angular momentum

$$\hat{J}^2 = J\,\hat{G}\,J = B^2(p)\; J^2, \tag{15.53}$$

is indeed a neutral element of the Lie–isotopic envelope, as the reader can verify.

EXPERIMENTAL TEST OF SANTILLI'S RELATIVITIES IN PARTICLE PHYSICS



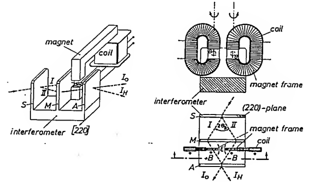


FIGURE 5: A central capability of Santilli' (1991d) relativities is that of representing the actual extended character of a particle, say, an oblate spheroidal ellipsoid, as well as all its

infinitely possible deformations. When applied to a hadron, Santilli's relativities therefore predict that their intrinsic magnetic moment can be altered (mutated) under sufficiently intense external forces.

Rigid bodies do not exist in the physical reality. As a result, the amount of deformation of the charge distribution of a hadron, say a neutron, under given external forces is open to scientific debate. But the existence of the deformation itself under sufficient external forces is beyond any credible doubt.

Once the deformability, even minimal, of the charge distribution of a neutron is admitted, the mutation of its intrinsic magnetic moment is a mere consequence of Maxwell's electrodynamics.

Intriguingly, these fundamental predictions of Santilli's relativities are confirmed in a preliminary way by series of experiments conducted by H. Rauch and his associates (see the reviews by Rauch (1981), (1983) and quoted experimental papers).

These experiments have been studied via the *operator* formulation of isotopic theories by Santilli (1981), (1989), Eder (1981) and (1983)), and others. However, it is important to show that the *classical* methods presented in this volume can already provide an approximate, yet quantitative and physically meaningful representations of the experimental data.

A neutron interferometers is essentially constituted by a neutron beam which is subjected to a coherent splitting into two branches via a perfect crystal, and then their recombination. The neutron beam is generally monochromatic, unpolarized and with high flux. The perfect crystal is generally given by a Si crystal with extremely low impurities which allows theachievement of angles of separation of the two branches sufficiently wide to permit experiments in one branch or in both.

In his experiments, Rauch used: a thermal neutron beam with a cross section of about $2 \times 1.4\ mm^2$; a characteristic wavelength of the crystal of 1.83 A°; about 1 cm of electromagnetic gap; and a magnetic field of the intensity of 7,496 G which is calibrated to produce two, complete and exact spin-flips, say, around the third axis ($\theta_3 = 720°$), for neutrons with their conventional magnetic moment

$$\mu_n = - 1.91304211 \pm 0.0000011 \quad 2\hbar/2m_p c_o. \qquad (1)$$

The experimenters filled up the electromagnet gap with Mu-metal sheets for the primary purpose of reducing stray fields. It is this latter rather accidental feature that renders the experiments truly fundamental, inasmuch as they *test the rotational symmetry of the neutrons under EXTERNAL*[36] *magnetic and nuclear interactions.*

The best available measures currently available are given by (Rauch (*loc. cit.*))

$$\theta_3 = 715.87° \pm 3.8°, \qquad (2a)$$

[36] A primary emphasis must be given to the external character of the target and, consequently of the fields because, if one considers the system neutron plus nucleus, no mutation is possible.

$$\hat{\theta}_3^{\max} = 719.67°, \quad \hat{\theta}_3^{\min} = 712.07°, \tag{2b}$$

It should be indicated that the above measures do not establish the violation of the conventional rotational symmetry because the deviation should be of the order of at least four times the statistical error to achieve a sufficient degree of confidence. Thus, to establish the value 715.87°, the error should be of the order of ± 1°.

Despite this unsettle nature, the implications of the above measures are intriguing because:

1) Measures (2) do not confirm the exact rotational symmetry for the neutrons in the open conditions considered, by indicating a conceivable violation of about 1%.

2) None of the median angles measured by Rauch coincide with 720°. On the contrary, all experiments show a median angle consistently lower than 720°, an occurrence called "angle slow-down effect" (Santilli (1981)).

3) The measurements of the intensity modulation for measures (2) do not confirm the exact rotational symmetry because the modulation curve does not appear to be an exact co-sinusoid,

as well as for other reasons.

Measures (2) are therefore valuable for the following reasons. First, the reader should keep in mind that the best possible origin for an angle $\hat{\theta}_3$ different than the expected 720° is the alteration of the magnetic moment (1), i.e., the mutation of this figure

$$\mu_n \Rightarrow \hat{\mu}_n, \tag{3}$$

In turn, such an alteration of the magnetic moment can best occur under the deformation of shape indicated earlier.

Furthermore, it is important to achieve a theoretical representation of Rauch's experiments as a basis for the future finalization of the numerical amount of mutation under given external forces.

Note that the true symmetry tested by Rauch at the particle level is the *spinorial* SU(2) symmetry, rather than the *rotational* O(3) symmetry. Nevertheless, the issue deserves an analysis, first, within the context of the rotational symmetry O(3), prior to a study within the covering SU(2) extension, in order to separate the rotational from the spinorial contribution, besides the classical from the quantum mechanical one.

Santilli (1988a), (1991d) therefore provided a representation of Rauch's experiments via his isogalilean relativities which, even though only classical and nonrelativistic, is nevertheless direct and quantitative, by therefore focusing the attention of the experimenters in the classical origin of this event.

Recall that there are no "contact interactions" between the neutron beam and the Mu-metal nuclei. Thus, we can effectively use the case of extended and deformable isoparticles under external *potential* forces only.

We conclude this section with the remark that the general, nonlinear, nonlocal isorotational symmetries $\hat{O}(3)$ on isospaces $\hat{E}(r,\hat{\delta},\hat{\mathfrak{R}})$ are the global

symmetries of the space component of the conventional and isotopic theories of gravitation.

$$T^*\hat{E}(r,\hat{\delta},\hat{\mathfrak{R}}):\ \hat{\mathfrak{R}} = \mathfrak{R}\hat{1},\quad \hat{1}_1 \equiv \hat{1}_2 \equiv \hat{1} = \mathrm{diag.}\,(\hat{\delta}^{-1}, \hat{\delta}^{-1}), \tag{4a}$$

$$\hat{\delta} = \mathrm{diag.}\,(b_1^2, b_2^2, b_3^2),\quad b_k = \text{constants} > 0. \tag{4b}$$

and the isoanalytic representations given in the text.

Santilli's treatment is independent from the explicit form of the external potential V(r) and is essentially based on the assumption that the external field implies the deformation of the shape

$$\delta = \mathrm{diag.}\,(1, 1, 1),\ V = 0 \ \Rightarrow\ \hat{\delta}' = \mathrm{diag.}\,(b'^2_1, b'^2_2, b'^2_3),\ V \neq 0, \tag{5}$$

under the conditions of being volume preserving,

Suppose that $T^*E(r,\delta,\mathfrak{R})$ is the conventional phase space of the neutron beams. Then, the isogalilean relativities uniquely follows by assuming the isospace

$$b_1^2 b_2^2 b_3^2 = b'^2_1 b'^2_2 b'^2_3, \tag{6}$$

Since we have at best a small deviation, it is reasonable to assume that the mutation of shape is also small. In first approximation, Santilli therefore assumes from data (2) that the deviation is of the order of

$$716^\circ/720^\circ \cong 0.9944, \tag{7}$$

which can be also assumed to be of the order of magnitude of the (evidently average) oblateness caused by the external nuclear fields.

Then, Santilli's purely classical and nonrelativistic treatment implies that mutation (5) for values (7) under condition (6), assumes the numerical values

$$\delta = \mathrm{diag.}\,(1,1,1),\ V = 0 \ \Rightarrow\ \hat{\delta}' = \mathrm{diag.}\,(1.0028, 1.0028, 0.9944),\ V \neq 0 \tag{8}$$

with consequential mutation of the magnetic moment of the order of 6×10^{-3}, i.e.,

$$\mu_n \approx -1.913\, e\hbar/2m_p c_o \ \Rightarrow\ \hat{\mu}_n \approx -1.902\, e\hbar/2m_p c_o, \tag{9}$$

which does indeed provide a first, approximate, but quantitative interpretation of Rauch's data (2).

Intriguingly, Santilli's isogalilean relativities are not only capable to represent measures (2), but also the *"angle slow-down effect"* (Santilli (1981)), namely, the fact that the median angles measured by the experimenters during the several years of the conduction of the tests have been consistently *lower* than the needed 720°.

In fact, mutated value (9) is *lower* than the original valuer (1), thus implying angles of spin-flips necessarily lower than 720° for a magnetic field of 7,946 G.

Moreover, the conventional rotational symmetry is evidently broken for values (2). Nevertheless, *Santilli's isogalilean relativities reconstruct the exact rotational symmetry for the deformed neutrons.* This is another aspect that warrants an identification, first, at the primitive Newtonian level, and then at the operator counterpart.

For this purpose, Santilli considers the subgroup of $\hat{G}(3.1)$ given by the isorotational symmetries $\hat{O}(3)$. As now familiar, the isotopes $\hat{O}(3)$ provide the form-invariance of all possible ellipsoidical deformations of the sphere, while being locally isomorphic to the conventional rotational symmetry $O(3)$. This establishes the reconstruction of the exact rotational symmetry for deformed charge distributions (8), of course, at the isotopic level.

However, the mechanism of such reconstruction deserves a deeper inspection because important for Rauch's experiments. For this purpose, Santilli considers the isorotation around the third axis, Eq s (15.42).

The reconstruction of the exact rotational symmetry is then based on mechanism (15.43) originating from the values b_1 and b_2 of Eq. (8) and Rauch's median angle (2), i.e.,

$$\hat{\theta}'_3 = b_1 b_2 \hat{\theta}_3 \Big|_{\hat{\theta}_3 = 716^\circ,\ b_1 = b_2 = 1.0028} \cong 720^\circ, \tag{12}$$

namely, *Santilli's geometric isospace $\hat{E}(r,\hat{\delta},\hat{\mathfrak{R}})$ reconstructs the angle $\theta'_3 = 720°$ needed for the exact rotational symmetry from* an *actual rotation of $\hat{\theta}_3 = 714°$ in our physical space $E(r,\delta,\mathfrak{R})$.*

In conclusion, Santilli's isogalilean relativities can:

a) directly represent the actual shape of the neutron;

b) directly represent all possible deformations of said shape caused by sufficiently intense external fields and/or collisions;

c) directly represent the consequential mutation of the intrinsic magnetic moment of the particle;

d) directly represent the "angle slow-down effect" because of the decreased value of the magnetic moment, and

e) reconstruct the exact rotational and Galilei symmetries at the more general isotopic level.

The advances permitted by Santilli's relativities over the conventional relativities are then incontrovertible.

Not surprisingly, the operator and relativistic treatments (see Santilli (1989a, b, c, d)) confirm in full the above classical and nonrelativistic results.

Santilli (1981b) therefore proposed the resolution of the issue via the repetition of

Rauch's experiments along the following primary lines:

PROPOSED EXPERIMENTS:

I) *REPEAT RAUCH'S EXPERIMENTS WITH THE BETTER ACCURACIES CURRENTLY AVAILABLE.* In fact, the reduction of the current error of about 80% would establish Santilli's mutation of the magnetic moment.

II) *REPEAT RAUCH'S EXPERIMENTS WITH A MULTIPLE OF TWO SPIN FLIPS.* Apparently, current neutron interferometers can reach up to fifty spin flips and more. Since the phenomenon of mutation is expected to be nonlinear, the increase of the number of spin flips is expected to produce a higher deviation, which could therefore resolve the issue even the with accuracy of measures (2).

III) *REPEAT RAUCH'S EXPERIMENTS WITH A HEAVIER NUCLEI.* In fact, the mutation is also expected to be nonlinearly dependent on the mass of the nuclei in the electromagnet gap. The use of metal heavier than that used so far could also resolve the issue.

In these experiments, *SANTILLI SUGGESTS THE MEASURE NOT ONLY OF THE MEDIAN ANGLE, BUT ALSO THE PLOTTING OF THE CURVE OF THE INTENSITY MODULATION AND OF ITS PHASE WITH THE POLARIZATION.*

The implications of Santilli's prediction of a mutation of the intrinsic magnetic moments of particles under sufficient external forces, are manifestly far reaching. We here mention the possibility of : finally resolving the vexing problem of the total nuclear magnetic moments mentioned in Sect. 13; implying, sooner or later, a revision of nuclear physics; permitting fundamentally new structure models of hadrons; etc.

The societal implications are also considerable. In fact, it is possible that the current inability to achieve the controlled fusion with a positive energy output is due precisely to the currently used Einsteinian representation of protons and neutrons as abstract, perennial and immutable points. Such a representation is evidently exact during the compression phase of the plasma. However, the representation becomes highly questionable at the instant of initiation of the fusion process precisely because of the activation at that instant of Santilli's mutation, this time, with short range, nonlinear, nonlocal and nonhamiltonian interactions.

At any rate, the experimental confirmation of Rauch's measures (2) would invalidate the very conception and design of the current magnetic bottle, trivially, because they are set on intrinsic magnetic moments (1) which would be altered at the very initiation of the fusion process.

This author therefore concludes the main text of this book with the hope that Santilli's relativities will indeed be developed by mathematicians and theoreticians in the field, and its experimental tests conducted in the near future. In the final analysis, ***we are referring to aspects of contemporary knowledge so fundamental, to dwarf, by comparison, several other, currently preferred, mathematical, theoretical and experimental research.***

APPENDIX I.A: LIE-ADMISSIBLE STRUCTURE OF HAMILTON'S EQUATIONS WITH EXTERNAL TERMS

No in depth knowledge of topic of this volume can be achieved without a study of the analytic, algebraic and geometrical structures underlying the equations originally submitted by Hamilton (1834) for the interior dynamical systems (1.1), those with external terms

$$\dot{r}_{ka} = \frac{\partial H(t, r, p)}{\partial p_{ka}} = p_{ka}/m_a, \tag{A.1a}$$

$$\dot{p}_{ka} = -\frac{\partial H(t, r, p)}{\partial r_{ka}} + F_{ka}, \tag{A.1b}$$

$$H = p_{ka}\, p_{ka} / 2m_a + V(r), \tag{A.1c}$$

$$F_{ka} = F^{NSA}{}_{ka}(t, r, p, \dot{p}, ...) + \int_\sigma d\sigma \mathfrak{F}^{NSA}{}_{ka}(t, r, p, \dot{p}, ...), \tag{A.1d}$$

$$k = 1, 2, 3\ (= x, y, z),\ a = 1, 2, ..., N$$

As one can see, the above equations verify a rather simple "direct universality" for the representation of all possible systems (1.1) in the coordinates of the experimenter, because the Hamiltonian H represents all local and potential forces, while the external terms F_{ka} represents all remaining nonlinear, nonlocal and nonhamiltonian forces.

However, in so doing, the Hamiltonian H is necessarily nonconserved (Sect. 10) and, for this reason, the equations characterize an open nonconservative system.

In this appendix we shall also review the algebraic structure of the most general possible nonautonomous Birkhoff's equation (7.11) in $T^*E(r,\delta,\mathfrak{R})$ with local coordinates $a = (a^\mu) = (r, p) = (r_k, p_k)$,

$$\dot{a}^{\mu} = \Omega^{\mu\nu}(t,a)\,[\frac{\partial B(t,a)}{\partial a^{\nu}} + \frac{\partial R_{\nu}(t,a)}{\partial t}],\quad \mu,\nu = 1, 2, ..., N, \tag{A.2a}$$

$$\Omega^{\mu\nu} = (|\Omega_{\alpha\beta}|^{-1})^{\mu\nu}, \tag{A.2b}$$

$$\Omega_{\mu\nu} = \frac{\partial R_{\nu}(t,a)}{\partial a^{\mu}} - \frac{\partial R_{\mu}(t,a)}{\partial a^{\nu}}, \tag{A.2c}$$

The algebraic structure of Eq.s (A.1) was identified, apparently for the first time in Santilli ((1967), (1968), (1969)). The studies were then continued in Santilli (1978a). A comprehensive presentation can be found in Santilli (1981a), including the identification of an underlying geometric structure and the extension of the results to Eq.s (A.2).

In this appendix we shall outline the algebraic properties of Eq.s (A.1) and (A.2), as well as point out in more details the reasons why the restriction of the studies of interior trajectories solely to the isotopic treatments is insufficient, and the additional use of the complementary Lie-admissible formulations is recommendable. Further properties will be briefly outlined in the subsequent appendices. The content of these appendices is essentially derived from the appendices of Santilli (1991b).

To begin, the conventional Poisson brackets [A,B] of Hamilton's equations without external terms are generalized for Eq.s (A.1) in a form, say A×B, which is explicitly given by

$$A\times B = [A,B] + \frac{\partial A}{\partial p_{ka}} F_{ka}. \tag{A.3}$$

PROPOSITION A.1. (loc. cit.): Brackets (A.3) of Hamilton's equations with external terms violate the conditions to characterize any algebra.

PROOF. Brackets (A.3) violate the right scalar and right distributive laws (5.1), i.e.,

$$\alpha\times(B\times C) = A\times(\alpha\times B) = (\alpha\times A)\times B, \tag{A.4a}$$

$$(A\times B)\times\alpha \neq A\times(B\times\alpha) \neq (A\times\alpha)\times B, \tag{A.4b}$$

and

$$(A + B)\tilde{x}C = A\tilde{x}C + B\tilde{x}C, \tag{A.5a}$$

$$A\tilde{x}(B + C) \neq A\tilde{x}B + A\tilde{x}C, \tag{A.5b}$$

As a result, they do not characterize an algebra as commonly intended in contemporary mathematics (Jacobson (1962).QED

In different terms, *in the transition from the contemporary Hamilton's equations to their original form with external terms, we have the loss, not only of the Lie algebras, but more precisely of all consistent algebraic structures.*

Exactly the same situation occurs for the quantum mechanical treatment of nonconservative forces via nonhermitean Hamiltonians $H_d \neq H_d^\dagger$ (Santilli (1978b)). In fact, under these conditions, the conventional Heisenberg's brackets among operators A, B, ... on a Hilbert space $\mathcal{H}$, [A,B] = AB − BA, over a complex field C are generalized into a form, say $A\tilde{x}B$, which is evidently defined by the new equations of motion

$$i\dot{A} = A\tilde{x}H_d = AH_d^\dagger - H_dA, \quad \hbar = 1, \tag{A.6}$$

Again, *the nonconservative Heisenberg's brackets $A\tilde{x}H$, not only lose the Lie algebra character of conventional quantum mechanics, but do not characterize any consistent algebra, because they violate the right scalar and right distributive laws,* as the reader is encouraged to verify.

This is not a mere mathematical occurrence, because it carries rather deep physical implications. For instance, the notion of *angular momentum* can be consistently defined in conventional (classical and quantum) Hamiltonian mechanics, and treated via its underlying Lie symmetry O(3).

In the transition to Hamilton's equations with external terms (A.1) and their operator counterpart (A.6), we have lost *all* Lie algebras, let alone that of the rotational symmetry. This has the direct consequence that, even though the use of angular momenta is kept for Eq.s (A.6) according to a rather widespread use in contemporary particle and nuclear physics, the reality is that the notion has lost all necessary background for its definition, let alone its quantitative treatment.

In fact, it would be inconsistent to use one product $A\tilde{x}H$ for the time evolution, and a *different product,* say, [A,H] for the characterization of physical quantities such as the angular momentum. This is due to the well known, ancient rule of dynamics whereby *the product of the algebra characterizing a given theory, whether classically or operationally, must coincide with that of the time evolution law.*

To put it explicitly, a statement to the effect that, say, a particle described

by Eq.s (A.6) has spin one, is not mathematically substantiated, because of the loss of any algebra for its treatment, and physically unfounded, because the spin of particles in open nonconservative conditions is ultimately unknown at this writing.

Exactly the same situation occurs for the nonautonomous Birkhoff's equations (Santilli (1981a)). In fact, Birkhoff's brackets $[A \hat{,} B]$ for the autonomous case (Sect. 8),

$$[A \hat{,} B] = \frac{\partial A}{\partial a^{\mu}} \Omega^{\mu\nu}(a) \frac{\partial B}{\partial a^{\nu}} \tag{A.7}$$

have to be generalized for Eq.s (A.2) in the form

$$A \odot B = [A \hat{,} B] + \frac{\partial A}{\partial a^{\mu}} \Omega^{\mu\nu} \frac{\partial R_{\nu}}{\partial t} \tag{A.8}$$

which again violate the right scalar and distributive laws.

Equivalently, one can say that for, the case of time-dependent R-functions, Birkhoff's equations can be expressed with the (2N+1)×(2N+1) contact tensor (Sect. 9) which, being odd-dimensional, do not admit a consistent contravariant (Lie) counterpart.

The reader should therefore be aware that *the isotopies of conventional relativities* are *inapplicable to the nonautonomous Birkhoff's equations, because of the loss of* a *consistent algebraic structure, let alone the loss of their Lie-isotopic character* (Santilli (loc. cit.)).

The above occurrences evidently create the problem of identifying the relativities which are directly applicable to open, nonconservative, nonautonomous, interior systems, such as oscillator with a time-dependent applied force, etc.

In turn, the above relativities cannot be identified without first reformulating Eq.s (A.1) and (A.2) in an analytically *identical* way (to avoid the alteration of the equations of motion) which is however admitting of a consistent algebraic structure.

This problem signals the birth of the Lie-admissible algebras in physics (Santilli (1967) [37]. In fact, on one side, the consistent brackets for Eq.s (A.1), say, (A,B), cannot be antisymmetric, to permit the representation of the time-rate-of-variation of the energy

∂H

[37] This is the quotation Santilli (1967) in the historical chart of the Estonian Academy of Sciences at the end of these appendices.

$$\dot{H} = (H,H) = \frac{}{\partial p_{ka}} F_{ka} = v_{ka}F_{ka}, \qquad (A.9)$$

while, on the other side, Lie algebras cannot be evidently abandoned, because they must be admitted as a particular case for null nonselfadjoint forces, i.e.

$$(A,B)_{|F_{ka}=0} \equiv [A,B]. \qquad (A.10)$$

This problem was originally studied in Santilli ((1967),(1968), (1969)) and then reinspected in Santilli (1978a), where it was pointed out that conditions (A.10) identify the so-called *general Lie-admissible algebras*.

According to Albert (1948) an algebra **U** with (abstract) elements a,b,c,...and (generally nonassociative, abstract) product ab over a field F

$$\mathbf{U}: \quad ab = \text{nonassociative} \qquad (A.11)$$

is called a *Lie-admissible algebra* (Sect. 5) when the attached algebra $\mathbf{U}^-$, which is the same vector space as **U**, but equipped with the product IS LIE,

$$\mathbf{U}^-: \quad [a,b]_{\mathbf{U}} = ab - ba, \qquad (A.12)$$

Evidently, all *associative algebras* **A** are Lie-admissible, resulting in the familiar Lie product $[a,b]_{\mathbf{A}} = ab - ba$, where now ab is associative.

All *Lie algebras* **L** with (abstract) product ab are also Lie-admissible, because $[a,b]_{\mathbf{L}} = 2[A,B]_{\mathbf{A}}$, where now ab is nonassociative. Thus, Lie algebras are contained in Lie-admissible algebras in a two-fold way, first, in the classification and, second, as the attached antisymmetric algebras.

The most general possible algebras of the type considered are by Santilli (1978a) the *general Lie-admissible algebras* when they verify no condition other than the Lie-admissibility law (A.12).

The first classical realization of the Lie-admissible algebras in physics was introduced in Santilli (1978a, c) and then worked out in more details in Santilli (1981a) Let A, B be (nonsingular, sufficiently smooth) functions in $\Re_t \times T^*E(r,\delta,\Re)$. Then the brackets

$$(A,B) = \frac{\partial A}{\partial a^{\mu}} S^{\mu\nu}(t,a) \frac{\partial B}{\partial a^{\nu}}, \qquad (A.13)$$

over the reals $\Re$ characterize a Lie-admissible algebra **U** when the attached

antisymmetric brackets

$$U^-: \quad [A,B]_U = (A,B) - (B,A) \tag{A.14}$$

are Lie, or, equivalently, when the attached antisymmetric tensor

$$S^{\mu\nu} - S^{\nu\mu} = \Omega^{\mu\nu} \tag{A.15}$$

is Birkhoffian.

Now, the direct way of writing brackets (A.3) in an algebraically consistent way is by introducing the tensor in $\Re_t{\times}T^*E(r\delta,\Re)$

$$S^{\mu\nu}(t,a) = \omega^{\mu\nu} + s^{\mu\nu}(t,a), \tag{A.16}$$

where $\omega^{\mu\nu}$ is the (totally antisymmetric) canonical Lie tensor (7.16), and $s^{\mu\nu}$ is the totally symmetric tensor

$$s = (s^{\mu\nu}) = \mathrm{diag.}\,(0, s), \quad s = F/(\partial H/\partial p) \tag{A.17}$$

The brackets (A,B), when written in form (A.13) with the S–tensor given by *symmetric* form (A.16), first of all, verify both right and left scalar and distributive laws, and, secondly, they characterize a Lie–admissible algebra because the attached brackets are Lie

$$(A,B) - (B,A) = 2\,[A,B], \quad S^{\mu\nu} - S^{\nu\mu} = 2\omega^{\mu\nu}. \tag{A.18}$$

Finally, the equations of motion are not altered when rewritten in terms of tensor (A.16), i.e.,

$$\dot{a}^{\mu} = S^{\mu\nu}\frac{\partial H(t,a)}{\partial a^{\nu}} = (a^{\mu}, H), \tag{A.19}$$

called *Hamilton–admissible equations* (Santilli (1981a)). In fact, we have

$$\dot{r}_{ka} = \partial H/\partial p_{ka}, \tag{A.20a}$$

$$\dot{p}_{ka} = -\partial H/\partial r_{ka} + s_{kajb}\,\partial H/\partial p_{jb} = -\,\partial H/\partial R_{ka} + F_{ka}. \tag{A.20b}$$

In particular, the brackets (A,B) preserve the CORRECT time-rate-of-variation of the Hamiltonian

$$\dot{H} = (H,H) = v_{ka}F_{ka}, \qquad (A.21)$$

as desired.

The regaining of a consistent mathematical structure carries rather intriguing physical implications.

As an example, *Eq.s (A.1) do not admit a consistent exponentiation into a finite group*. On the contrary, when written in their equivalent Lie-admissible form (A.19), they can be easily exponentiated into the form

$$a' = \{ e_{|A}^{t \, ^oS^{\mu\nu} \partial_\nu H)(\partial_\mu)} \} a, \qquad (A.22)$$

In particular, *the above structure leaves invariant the equations of motion.* In fact, from a general property of vector-fields on manifolds, we have

$$\Gamma(t,a') = \{ e_{|A}^{t \, ^oS^{\mu\nu}(\partial_\nu H)(\partial_\mu)} \} \Gamma(t,a) = \Gamma(t,a'), \qquad (A.23)$$

For this reason, structures of type (A.22) constitutes an intriguing generalization of the notion of Lie-Santilli symmetry (Sect. 9) known as a *Lie-admissible symmetry* (Santilli (*loc. cit.*)).

The physical differences with the conventional approach are, however, rather deep. In fact, *the conventional Lie and Lie-isotopic symmetries represent the conservation of the energy and other quantities.* In the more general case under consideration here, we can say that *the broader Lie-admissible symmetry characterized by the Hamiltonian as generator represent the time-rate-of-variation of the energy*

$$\dot{H} = H(t,a) - \{ e_{|A}^{t \, ^oS^{\mu\nu}(\partial_\nu H)(\partial_\mu)} \} H(t,a) = v_{ka}F_{ka}. \qquad (A.24)$$

Moreover, exponentiation (A.19) admits the following explicit form

$$\{ e^{t^o S^{\mu\nu} (\partial_\nu H)(\partial_\mu)} |_A \} A = A + t^o (A,H)/1! + t^{o2}((A,H),H)/2! + \ldots \quad (A.25)$$

namely, *symmetries (A.19) admit non-Lie, Lie-admissible algebras in the neighborhood of the identity.* This signals the possibility of generalizing the entire Lie's and Lie-isotopic theories in a yet more general Lie-admissible theory (Santilli (1978a), (1982a)).

The mathematical and physical covering character of the Lie-admissible formulations over the Lie-isotopic and Lie formulations is then evident.

By recalling that the symmetry characterized by the Hamiltonian as generator is the time component of the Galilei and of the Galilei-isotopic relativities, symmetry (A.23) can then be considered as the time component of conceivable, still more general relativities, tentatively called *Lie-admissible relativities* (*loc. cit.*) for open nonconservative systems, in which the form-invariance characterizes, this time, the time-rate-of-variation of Galilean quantities. The understanding is that the studies on Lie-admissibility are considerably less advanced than the corresponding Lie-Santilli theories, and so much remains to be done.

The identification of the algebraic structure of the nonautonomous Birkhoff's equations (A.2) is now easy (*loc. cit.*). Introduce the generalized tensor

$$S^{\mu\nu}(t,a) = \Omega^{\mu\nu}(a) + T^{\mu\nu}(t,a), \quad (A.26)$$

where $\Omega^{\mu\nu}$ is the (totally antisymmetric) Birkhoff's tensor (A.2b), and $T^{\mu\nu}$ is given by the totally symmetric form

$$T = (T^{\mu\nu}) = \text{diag.}(0,\sigma), \quad \sigma = (\partial_t R) / (\partial_\mu B). \quad (A.27)$$

Then, the generalized brackets

$$(A\hat{,} B) = \frac{\partial A}{\partial a^\mu} S^{\mu\nu}(t,a) \frac{\partial B}{\partial a^\nu}, \quad (A.28)$$

are algebraically consistent and Lie-admissible, as one can see, thus resulting in the generalized equations

$$\dot{a}^{\mu} = \hat{S}(t, a) \frac{\partial B(t, a)}{\partial a^{\nu}}, = (\Omega^{\mu\nu}(a) + \tau^{\mu\nu}(t, a)] \frac{\partial B(t, a)}{\partial a^{\nu}}, \tag{A.29}$$

called *Birkhoff-admissible equations* (Santilli (1978a), (1981a)) and which evidently constitute a covering of both Birkhoff's and Birkhoff-Santilli equations.

In particular, the transition from brackets (A.13) to (A.28) is an example of *Lie-admissible isotopies* (Sect. 5).

For further studies, we refer the interested reader to Santilli (1981a), where one can see the elements for a further generalization of Birkhoffian mechanics into a covering discipline, tentatively called *Birkhoffian-admissible mechanics.*

The operator counterpart of Hamilton-admissible equations (A.16) was achieved in Santilli (1978b). We shall here briefly outline it, because the operator Lie-admissible equations possesses considerable guidance value in the study of the abstract Lie-admissible formulations.

The most salient physical difference in the transition from closed-isolated-stable systems to open-nonconservative-unstable systems is the appearance of *irreversibility*, i.e., the lack of invariance of physical processes under time reversal. As an example, the trajectory of Jupiter in the Solar system is manifestly reversible, while the trajectory of a satellite penetrating Jupiter's atmosphere is manifestly irreversible. Corresponding similar situations occur at the particle level.

By following Santilli (1991b), consider then the *forward direction in time*, and denote it with the symbol " > ". Let ξ be the conventional enveloping operator algebra of quantum mechanics with operators A, B, ... and trivial associative product AB on a Hilbert space $\mathcal{H}$ over the field of complex numbers C.

Introduce the isotope $\xi^{>}$ of ξ (Sect. 5) describing the motion forward in time

$$\xi^{>}: \quad A{>}B = AT^{>}B, \tag{A.30}$$

which is characterized by a nowhere null and sufficiently smooth, but *nonhermitean* operator $T^{>}$, with *isounit for motion forward in time*

$$1^{>} = (T^{>})^{-1}, \tag{A.31a}$$

$$1^{>}{>}A = A{>}1^{>} = A, \qquad \forall\, A \in \xi^{>}, \tag{A.31b}$$

Introduce now the isotope ${}^{<}\xi$ for motion backward in time, which is denoted with the symbol " < ",

$$^{<}\xi: \quad A<B = A\,^{<}TB, \tag{A.32}$$

characterized by a different isotopic element $^{<}T \neq T^{>}$, with *isounit for motion backward in time*

$$^{<}I = (^{<}T)^{-1}, \tag{A.33a}$$

$$^{<}I<A = A<\,^{<}T, \tag{A.33b}$$

Finally, assume that the forward description via envelope $\xi^{>}$ is the time reversal of the backward one $^{<}\xi$. This can be done by assuming

$$I^{>} = (^{<}I)^{\dagger}. \tag{A.34}$$

LEMMA A.1 (loc. cit.):The axiomatic structure of irreversibility from the algebraic viewpoint can be expressed via isoassociative algebras with two different isounits $I^{>} = (^{<}I)^{\dagger} \neq {}^{<}I$, and related isofields, one for the motion forward in time $I^{>}$ and the other for the motion backward in time $^{<}I$.

It is an instructive exercise for the reader interested in learning the techniques of this volume to prove that structures (A.30)-(A.34) are indeed invariant under isotopy and, thus possess an axiomatic character.

Lemma A.1 is of particular guidance value in studying abstract problems, i.e., the identification of the generalization of the Riemannian geometry needed for the Lie-admissible formulations (Appendix C).

Under envelopes $\xi^{>}$ and $^{<}\xi$, the time evolution is given in infinitesimal form by

$$i\dot{A} = (A\,\hat{,}\,B) = A<H - H>A = A\,^{<}TH - HT^{>}A, \quad \hbar = 1, \tag{A.35}$$

with finite version

$$A(t) = {}^{<}I\,\{e|_{\xi}{}^{-it<H}\} < A(0) > \{e|_{\xi}{}^{iH>t}\}\,I^{>}, \tag{A.36}$$

which were proposed, apparently for the first time, in Santilli (1978b), p. 746.

It is easy to see that Eq.s (A.35) are Lie-admissible. In fact, their attached

antisymmetric bracket is precisely the brackets of the Lie–Santilli time evolution in operator form (Sect. 6)

$$i\dot{A} = [A \hat{,} B] = ATB - BTA, \qquad \text{(A.37a)}$$

$$T = {}^{<}T + T^{>}. \qquad \text{(A.37b)}$$

This shows again, this time at the operator level, the complementarity of the Lie–Santilli and Lie–admissible formulations.

In particular, structure (A.36) is an operator realization of the Lie–admissible groups (A.22).

It should be stressed that, *by no means Eq.s (A.35) alter the physical content of conventional nonconservative systems (A.6). In fact, Eq.s (A.35) merely provide the identical reformulation of the systems but, this time, in an algebraically consistent form.*

In fact, the nonhermitean Hamiltonians H_d of current use in physics are generally the sum of a Hermitean term H and a dissipative nonhermitean term

$$H_d = H + H_{diss.}. \qquad \text{(A.38)}$$

The desired, algebraically consistent, but physically identical reformulation of systems (A.6) is then given by (Santilli (*loc. cit.*))

$$H_d^{\dagger} = {}^{<}TH, \qquad H_d = HT^{>}, \qquad \text{(A.39a)}$$

$$i\dot{A} = AH_d^{\dagger} - H_dA \equiv A{<}H - H{>}A. \qquad \text{(A.39b)}$$

where now the *Hermitean* operator H evidently represents the *nonconserved* energy.

The similarities of the above operator formulation with the corresponding, classical, Birkhoff's and Birkhoff-admissible formulations, are remarkable, thus establishing the applicability of the complementary Lie–Santilli and Lie–admissible formulations at both the classical and operator level.

Thus, by no means, the isotopic relativities presented in the main text of this volume can be considered as the final relativities, because of the expected existence of the more general Lie–admissible relativities.

APPENDIX I.B: SYMPLECTIC-ADMISSIBLE GEOMETRY

As stressed by Santilli throughout this studies, physical theories in general, and relativities in particular, are a symbiotic expression of analytic, algebraic and geometric formulations.

The analytic and algebraic structures of the Birkhoff-admissible equations (A. 29) have been indicated earlier. It may therefore be of some value for the interested reader to outline the geometric structure of the Lie-admissible algebras.

This problem was studied in Santilli (1978a) and (1981a), and resulted in the submission of a new geometry under the name of *symplectic-admissible geometry.*

Recall that the direct geometric structure underlying Birkhoff's brackets (Sect. 7) in $T^*E(r,\delta,\Re)$ with the now usual unified notation $a = (a^\mu) = (r, p)$, $\mu = 1, 2, ..., 2n$,

$$[A \,\hat{,}\, B] = \frac{\partial A}{\partial a^\mu}\, \Omega^{\mu\nu}(a)\, \frac{\partial B}{\partial a^\nu}\,, \tag{B.1}$$

is the symplectic geometry also on $T^*E(r,\delta,\Re)$ characterized by the exact, symplectic, Birkhoffian two-forms

$$\Omega = \tfrac{1}{2}\, \Omega_{\mu\nu}(a)\, da^\mu \wedge da^\nu, \tag{B.2}$$

where the algebraic-contravariant and geometric-covariant tensors are interconnected by the familiar rule

$$\Omega^{\mu\nu} = (\,|\Omega_{\alpha\beta}|^{-1})^{\mu\nu}. \tag{B.3}$$

In the transition to the Birkhoff-isotopic brackets on isospaces $T^*\hat{E}_2(r,\hat{\delta},\hat{\Re})$ with isounit $\hat{1}_2$ (Sect. 8),

$$[A \,\hat{,}\, B] = \frac{\partial A}{\partial a^\mu}\, \Omega^{\mu\alpha}(a)\, \hat{1}_{2\alpha}{}^{\nu}(a)\, \frac{\partial B}{\partial a^\nu}\,, \tag{B.4}$$

we have the transition to the isosymplectic geometry (Sect. 9) characterized by the isoexact two–isoform

$$\hat{\Omega} = \tfrac{1}{2}\hat{1}_\mu{}^\alpha(a)\,\Omega_{\alpha\nu}(a)\,\hat{d}a^\mu \,\hat{\wedge}\, \hat{d}a^\nu \quad , \tag{B.5}$$

where, again, the algebraic and geometric tensors are interconnected by the rule

$$(\Omega^{\mu\alpha}\,\hat{1}_{2\alpha}{}^\nu) = (\hat{1}_{2\alpha}{}^\rho\,\Omega_{\rho\beta})^{-1}. \tag{B.6}$$

The problem of the geometry underlying the Birkhoff–admissible brackets (B.28), i.e.,

$$(A \,\hat{,}\, B) = \frac{\partial A}{\partial a^\mu}\,\hat{S}^{\mu\nu}(t,a)\,\frac{\partial B}{\partial a^\nu}\ , \tag{B.7a}$$

$$\hat{S}^{\mu\nu} = \Omega^{\mu\nu}\ T^{\mu\nu}, \tag{B.7b}$$

$$\Omega^{\mu\nu} = -\ \Omega^{\nu\mu}, \tag{B.7c}$$

$$T^{\mu\nu} = T^{\nu\mu}, \tag{B.7d}$$

was resolved via the introduction of a geometry more general than the symplectic and the isosymplectic ones.

We cannot possibly review these studies in detail here. Nevertheless an outline of the central ideas may be of some volume for the interested reader.

The first point to realize is that *the symplectic geometry and related exterior calculus, whether in their conventional or isotopic formulations, are intrinsically unable to characterize the Lie–admissible* algebras.

This is due to the fact that the calculus of exterior forms is essentially *antisymmetric* in the indices, while the Lie–admissible tensors $\hat{S}^{\mu\nu}$ are not, and the same occurs for the covariant counterpart

$$\hat{S}_{\mu\nu}(t,a) = (\,\|\,\hat{S}^{\alpha\beta}\|^{-1})_{\mu\nu} \neq \pm\,\hat{S}_{\nu\mu} \tag{B.8}$$

In fact, the construction of a conventional exterior two–form with the above tensor implies the reduction

$$\hat{S}_{\mu\nu}\, da^{\mu} \wedge da^{\nu} = \tfrac{1}{2}\, \hat{\Omega}_{\mu\nu}\, da^{\mu} \wedge da^{\nu}, \tag{B.9}$$

namely, the symplectic geometry automatically eliminates the symmetric component of the S-tensor, thus characterizing only its Lie content.

The main idea of Santilli's symplectic-admissible geometry is that of generalizing the conventional exterior calculus, say, of two differentials

$$da^{\mu} \wedge da^{\nu} = - da^{\nu} \wedge da^{\mu}, \tag{B.10}$$

into a more general calculus of differentials da^{μ} and da^{ν}, called *exterior-admissible calculus*, which is based on a product, say $\odot$ which is neither totally symmetric nor totally antisymmetric, but such that its antisymmetric component is the conventional exterior one,

$$da^{\mu} \odot da^{\nu} = da^{\mu} \wedge da^{\nu} + da^{\nu} \times da^{\nu}, \tag{B.11a}$$

$$da^{\mu} \wedge da^{\nu} = - da^{\nu} \wedge da^{\mu}, \tag{B.11b}$$

$$\partial a^{\mu} \times \partial a^{\nu} = \partial a^{\nu} \times \partial a^{\mu}, \tag{B.11c}$$

This allows the introduction of the *exterior-admissible forms* via the sequence

$$\hat{S}_0 = \phi(a), \tag{B.12a}$$

$$\hat{S}_1 = \hat{S}_{\mu}\, da^{\mu}, \tag{B.12b}$$

$$\hat{S}_2 = \hat{S}_{\mu\nu}\, da^{\mu} \odot da^{\nu}, \tag{B.12c}$$

....

The exact *exterior-admissible forms* are then given by

$$\hat{S}_1 = d\hat{S}_0 = \frac{\partial \phi}{\partial a^{\mu}}\, da^{\mu}, \tag{B.13a}$$

∂A_{ν}

$$\hat{\ }_2 = d\hat{S}_1 = \frac{}{\partial a^{\mu}} \, da^{\mu} \odot da^{\nu}, \tag{B.13b}$$

....

The calculus of exterior-admissible forms can indeed characterize the Lie-admissible algebras in full, because they characterize, not only the antisymmetric component of the Lie-admissible algebras, but also their symmetric part, via the two-forms

$$\hat{S}_2 = \hat{S}_{\mu\nu}(t,a)\, da^{\mu} \odot da^{\nu} =$$

$$= \hat{\Omega}_{\mu\nu}(a)\, da^{\mu} \wedge da^{\nu} + \hat{\tau}_{\mu\nu}(t, a)\, da^{\mu} \times da^{\nu}, \tag{B.14}$$

Structures (B.14) were called by Santilli (*loc. cit.*) *symplectic-admissible two-forms* because their antisymmetric component is symplectic, in a way fully parallel to the property whereby the antisymmetric part of the Lie-admissible algebras is Lie. Spaces $T^{*}E(r,\delta,\Re)$ when equipped with two-form (B.16) were called *symplectic-admissible manifolds* and the related geometry was called *symplectic-admissible geometry.*

As incidental comments, note that the dependence on time appears only in the symmetric part, as needed for consistency in the symplectic component. Also, under inversion (B.8), we generally have

$$(\hat{\Omega}_{\mu\nu}) \neq (\Omega^{\alpha\beta})^{-1}, \quad (\hat{\tau}_{\mu\nu}) \neq (\tau^{\alpha\beta})^{-1}, \tag{B.15}$$

which is a rather intriguing feature of the generalized geometry here considered, whereby the symplectic content of a Lie-admissible tensor is more general than the symplectic counterpart of the antisymmetric component of a Lie-admissible tensor (see Santilli (*loc. cit.*) for details).

The most salient departure of the exterior-admissible calculus from the exterior calculus in its conventional or isotopic formulation (Sect. 9) is that the Poincare' Lemma no longer holds, i.e., for exact symplectic-admissible two-forms we have

$$\hat{S}_2 = d\hat{S}_1, \tag{B.16a}$$

$$\hat{d}\hat{S}_2 = d(d\hat{S}_1) \neq 0. \tag{B.16b}$$

In actuality, within the contest of the exterior-admissible calculus, the Poincaré Lemma is generalized into a rather intriguing geometric structure which evidently admits the conventional Lemma as a particular case when all symmetric components are null.

The geometric understanding of the Lie-Santilli algebras requires the understanding that *the validity of the Poincaré Lemma within the context of the isosymplectic geometry is a necessary condition for the representation of the conservation of the total energy under nonhamiltonian internal forces*, as studied in the main sections of this volume.

By the same token, the geometric understanding of Santilli's more general Lie-admissible formulations requires the understanding that *the lack of validity of the Poincaré Lemma within the context of the symplectic-admissible geometry is a necessary condition for the representation of the nonconservation of the energy of an interior dynamical system.*

APPENDIX I.C: RIEMANNIAN-ADMISSIBLE GEOMETRY

According to Santilli, there is little doubt that future historians will consider our contemporary studies in gravitations as being in their first infancy.

Among a rather large number of problems that remain to be solved in gravitation, a further open problem is the representation of the dichotomy constituted by the time-reversible exterior dynamics with a clearly irreversible interior behavior.

This is majestically illustrated, e.g., by Jupiter (Figure 1) whose center-of-mass trajectory in the solar system is reversible, while the interior dynamics is manifestly irreversible.

It is at this point that the dual use of Santilli's Lie-isotopic and Lie-admissible formulations becomes useful. In fact, the *Lie-isotopic formulations are ultimately reversible in their structure*, because they provide a global treatment of nonhamiltonian systems via *Hermitean* isounits. By comparison, *the Lie-admissible formulations are intrinsically irreversible even when the Hamiltonians are reversible.*

Santilli refers to formulations that are structurally reversible or irreversible, rather than the achievement of reversibility or irreversibility via the selection of suitable Hamiltonians. In fact, Lie-isotopic formulations are irreversible irrespective of the selected Hamiltonian.

The dual representation of reversible center-of-mass-trajectories *versus* irreversible interior dynamics, is then permitted by the complementarity of the

Lie–isotopic and Lie–admissible formulations via inter-relations of type (A.37).

Note the necessity of the Lie–isotopic formulations for this complementarity. In fact, reversible, conventionally Lie formulations for the global–exterior description are not compatible with irreversible, Lie–admissible, interior descriptions because their attached Lie algebra is of Lie–isotopic character, as clearly expressed by Eq.s (A.37).

A first characterization of irreversibility was provided in Appendix A, via different isounits for motion forward $1^>$ and backward $^<1$ in time. A further approach to irreversibility will be provided in Appendix D via the notions of inequivalent right and left isorepresentations.

In this appendix we indicate a conceivable generalization of the Riemannian geometry, under the name of *Riemannian–admissible geometry*, originally submitted in (Santilli (1991b)) which provide an irreversible description of interior gravitation in a way compatible with the reversible description provided by the *Riemannian–isotopic geometry* of Sect. 11.

In Sect. 10 we introduced the notion of *affine–admissible manifolds* as the manifolds $^<M^>(x, {}^<\Re^>)$ which possess the same dimension, local coordinates and continuity properties of a conventional affine manifold $M(x,\Re)$, but are defined over an isofield $^<\Re^>$ with two different isounits $1^>$ and $^<1$ for the modular–isotopic action to the right and to the left, respectively

$$x'^> = A>x = AT^>x, \quad 1^> = (T^>)^{-1}, \tag{C.1a}$$

$$^<x' = x<A = x^<TA, \quad ^<1 = (^<T)^{-1}, \tag{C.1b}$$

$$1^> = (^<1)^\dagger. \tag{C.1c}$$

DEFINITION C.1 (loc. cit.): A "Riemannian–admissible manifold" is an isoriemannian manifold (Definition 11.1) with inequivalent isomodular actions to the right (forward) and to the left (backward), here denoted with $^<R^>(x, {}^<g^>, {}^<\Re^>)$, *namely, a manifold characterized by the "isometrics for motions forward and backward in time"*

$$g^> = T^>(x, \dot{x}, \ddot{x}, ...)\, g(x), \tag{C.2a}$$

$$^<g = {}^<T(x, \dot{x}, \ddot{x}, ...)\, g(x), \tag{C.2b}$$

$$T^{>} = ({}^{<}T)^{-1}, \qquad \text{(C.2c)}$$

and equipped with two nonequivalent, nonsymmetric, isoaffine connections, one for the modular-isotopic action to the right (forward) and the other to the left (backward), called "Christoffel-admissible symbols of the first kind for motions forward and backward in time",

$$\Gamma^{>}{}_{1hlk} = \tfrac{1}{2}\left(\frac{\partial g^{>}{}_{kl}}{\partial x^h} + \frac{\partial g^{>}{}_{lh}}{\partial x^k} - \frac{\partial g^{>}{}_{hk}}{\partial x^l}\right) \neq \Gamma^{>}{}_{1klh} \qquad \text{(C.3a)}$$

$${}^{<}\Gamma_{1hlk} = \tfrac{1}{2}\left(\frac{\partial {}^{<}g_{kl}}{\partial x^h} + \frac{\partial {}^{<}g_{lh}}{\partial x^k} - \frac{\partial {}^{<}g_{hk}}{\partial x^l}\right) \neq {}^{<}\Gamma_{1klh} \qquad \text{(C.3b)}$$

with corresponding "Christoffel-admissible symbols of the second kind"

$$\Gamma^{>}{}_{2}{}^{i}{}_{hk} = g^{>ij}\,\Gamma^{>}{}_{1hjk} = \Gamma^{>}{}_{2}{}^{i}{}_{kh} \qquad \text{(C.4a)}$$

$${}^{<}\Gamma_{2}{}^{i}{}_{hk} = {}^{<}g^{ij}\,{}^{<}\Gamma_{1hjk} = {}^{<}\Gamma_{2}{}^{i}{}_{kh} \qquad \text{(C.4b)}$$

where the capability for an isometric of raising and lowering the indices is understood (as in any affine space), and

$$g^{>ij} = |(g^{>}{}_{rs})^{-1}|^{ij}, \qquad \text{(C.5a)}$$

$${}^{<}g^{ij} = |({}^{<}g_{rs})^{-1}|^{ij}. \qquad \text{(C.5b)}$$

Santilli's "Riemannian-admissible geometry" is the geometry of spaces ${}^{<}\mathfrak{R}^{>}(x, {}^{<}g^{>}, {}^{<}\mathfrak{R}^{>})$.

The construction of the Riemannian-admissible geometry can be done via the appropriate generalization of the isoriemannian geometry presented in Sect. 11, with particular reference to the isoconnection which, besides being different for

the right and left modular-isotopic action, are now necessarily *nonsymmetric.*

Comparison of the above setting with that of Proposition A.1 and D.1 then yields the following

PROPOSITION C.1 (loc. cit.): An axiomatization of irreversibility in interior gravitation (only) is provided by inequivalent modular-isotopic actions to the right (forward in time) and to the left (backward in time) with necessarily nonsymmetric isoaffine connections.

Regrettably, we cannot study the Riemannian-admissible geometry in the necessary details to avoid a prohibitive length of this volume. It is however hoped that geometers in the field will indeed develop this new geometry for, in the final analysis, it is so general to encompass and include as particular cases all generalized geometries presented in this monograph.

The first generalization of Einstein's gravitation with a Lie-admissible structure was achieved by Gasperini (1983) in the full spirit of the formulations: to represent interior, nonconservative, irreversible trajectories, as well as a covering of the Lie-isotopic lifting of Einstein's gravitation on conventional manifolds (Gasperini (1984a,b,c)).

Additional important gravitational studies of Lie-admissible type were conducted by Adler (1978), Jannussis (1986), Gonzalez-Diaz (1986), Nishioka ((1985), (1987)), and others.

APPENDIX I.D: GENOREPRESENTATIONS

A deep understanding of Santilli's Lie-isotopic and Lie-admissible algebras cannot be reached without an understanding of the structure of their representation theories. In turn, the latter have well known, profound implications in physics, inasmuch as they characterize the notion of particle reviewed in the next appendix.

Santilli's Lie-isotopic and Lie-admissible formulations imply an intriguing sequence of generalizations of the representation theory along the following main lines:

A) REPRESENTATION THEORY OF LIE ALGEBRAS: characterized by one-sided, left or right, modular representations, generally called *"representations"*;

B) REPRESENTATION THEORY OF LIE-ISOTOPIC ALGEBRAS: characterized by one sided, left or right modular-isotopic representations, called

"isorepresentations" [38]; and

C) REPRESENTATION THEORY OF LIE-ADMISSIBLE ALGEBRAS: characterized by two-sided, left and right, modular-isotopic representations, called *two -sided isobirepresentations* , or *genorepresentations* [38] for short, first introduced in Santilli (1979);

By following the presentation in Santilli (1991b), consider a nonassociative algebra U over a field F. The right and left multiplications in U (Albert (1963), Schafer (1966)) are given by the following linear transformations of U onto itself as a vector space

$$R_x:\ a \Rightarrow ax, \quad \text{or} \quad aR_x = a_x, \tag{D.1a}$$

$$L_x:\ a \Rightarrow xa, \quad \text{or} \quad aL_x = xa, \tag{D.1b}$$

for all a, x e U, and verify the following general properties

$$(a\alpha)R_x = (a\alpha)x = a(\alpha x), \quad \text{or} \quad \alpha R_x = R_{\alpha x}, \tag{D.2a}$$

$$aR_{(x+y)} = a(x+y) = aR_x + aR_y = a(R_x + R_y)$$

$$\text{or } R_{(x+y)} = R_x + R_y \tag{D.2b}$$

with evident similar properties for the left multiplications L_x.

When the algebra is associative, we have the additional properties

$$a(xy) = (ax)y, \text{ or } aR_{xy} = aR_xR_y, \text{ or } R_{xy} = R_xR_y, \tag{D.3a}$$

$$(xy)a = x(ya), \text{ or } aL_{xy} = aL_xL_y, \text{ or } L_{xy} = L_xL_y. \tag{D.3b}$$

The above properties imply that the mapping $a \Rightarrow R_a$ ($a \Rightarrow L_a$) is a homomorphism (antihomomorphism) of A into the associative algebra V(A) of all linear transformations in A. Thus, they provide a *right representation* $a \Rightarrow R_a$ or a *left representation* $a \Rightarrow L_a$, respectively, of A, also called left or right $Hom^A{}_F(V_T)$,

[38] The algebraic meaning of Santilli's (1978a) *"isotopic and genotopic mapping"* was reviewed in Figure 1 of Sect. 10. These terms are now used to indicate the *"preservation"* and *"alteration"*, respectively, of the axiomatic structure of the representation theory of Lie algebras.

for $T = R, L$.

If the algebra A contains the identity I, we have a one-to-one *(or faithful) representation* because $R_a = R_b$ implies $I\ R_a = I\ R_b$ which can hold iff $a = b$. When the space L is the algebra A itself, we have the so-called *adjoint (or regular) representation.*

In the case of nonassociative algebras, the mapping $a \Rightarrow R_a$ is no longer a homomorphism, and this illustrates the reason for the study of the representation theory of Lie algebras via that of the underlying universal enveloping associative algebra, as done in the mathematical literature (see, e.g., Jacobson (1962)), but generally not in the physical literature.

Consider now an isoassociative algebra $\hat{A}$ over an isofield $\hat{F}$ with isounit $\hat{1}$ and isoassociative product $a*b$. Introduce the *right and left isomultiplications*

$$\hat{R}_x: a \Rightarrow a*x, \qquad \text{or} \quad a*\hat{R}_x = a*x, \tag{D.4a}$$

$$\hat{L}_x : a \Rightarrow x*a, \qquad \text{or} \quad a*\hat{L}_x = x*a, \tag{D.4b}$$

for all $a \in A$. It is then easy to see that properties () are lifted into the forms

$$\alpha*\hat{R}_x = \hat{R}_{\alpha*x}, \quad \hat{R}_{(x+y)} = \hat{R}_x + \hat{R}_y, \tag{D.5a}$$

$$\hat{R}_{x*y} = \hat{R}_x*\hat{R}_y, \quad \hat{1}*\hat{R}_a = \hat{1}*\hat{R}_b \quad \Rightarrow \quad a = b, \tag{D.5b}$$

with similar properties for the left isomultiplications.

It is easy to see that the mapping $a \Rightarrow R_a$ characterizes a *right, faithful, isorepresentation* of $\hat{A}$ in the isoassociative algebra $\hat{V}(\hat{A})$ of isolinear transformations of A$\Re$, and denoted $\mathrm{Hom}^{\hat{A}}{}_{\hat{F}}(\hat{V}_{\hat{R}})$, with similar results holding for the left isorepresentations.

The nontriviality of the isotopy is made clear by the following

LEMMA D.1 (Santilli (1991b): Isorepresentations of isoassociative algebras $\hat{A}$ over an isofield $\hat{F}$ are isolinear in $\hat{V}$ but generally nonlinear in V.

Namely, *the transition from Lie algebras to Lie-Santilli algebras generally implies the transition from linear and local to nonlinear and nonlocal representations.*

A *module* of an algebra U over a field F, also called *U-module*, (Schafer (1966)) is a linear vector space V over F together with a mapping $U{\times}V \Rightarrow V$ denoted

with the symbol $(a,v) \Rightarrow av$ which verifies the distributive and scalar rules

$$a(v + t) = av + at, \quad (a + b)v = av + bv, \tag{D.6a}$$

$$\alpha(a,v) = (\alpha a,v) = (a,\alpha v), \tag{D.6b}$$

as well as all the axioms of U, for all $a,b \in U$, $v,t \in V$, and $a \in F$.

The mappings $a \Rightarrow R_V = av$ and $a \Rightarrow L_V = va$ clearly show that the space V is a left and right U-module.

The above notion of module implies only one action, e.g., that to the right. In order to reach a two-sided action, consider an algebra U over a field F. Let V be a vector space over F. Introduce the direct sum $S = U \oplus V$ in such a way that S is an algebra verifying the same axioms of U while V is a two sided ideal of S. This can be done as follows (see, e.g., Schafer (1966)):

1) retain the product of U;

2) introduce a left and a right composition av and va, for all elements $a \in U$ and $v \in V$ which verify all axioms of U (including the and right and left scalar and distributive laws); and

3) to complete the requirement that V is an ideal of S, assume $vt = tv = 0$ for all elements of V.

When all the above properties are verified, V is called a *two-sided, left and right module*, or a *bimodule* of U, and the algebra S is called a *split null extension* of U (*loc. cit.*).

Bimodules clearly provide a generalized, left and right representation theory of all algebras, whether associative or nonassociative. It is important to understand why bimodules are not needed for the representation theory of Lie algebras (i.e., for the conventional notion of particles) and of Lie-isotopic algebras (i.e., for Santilli's isoparticle outlined in the main text), but they become essential for the covering Lie-admissible algebras.

A *bimodule V of a Lie algebra* L or *Lie-bimodule* (Santilli (1979a)) is characterized by left and right compositions av and va, $a \in L$, $v \in V$, verifying the properties

$$av = -va, \tag{D.7a}$$

$$v(ab) = (va)b - (vb)a, \tag{D.7b}$$

which can be identically expressed via the left and right multiplications

$$L_a = -R_a, \tag{D.8a}$$

$$R_{ab} = R_a R_b - R_b R_a, \tag{D.8b}$$

The mappings $a \Rightarrow R_a$ and $a \Rightarrow L_a$ then provide a *left and right representation*, or a *birepresentation*, of the Lie algebra L over the bimodule V as a $\mathrm{Hom}^{L}{}_{F}(V_R, V_L)$.

However, owing to property (D.8a), the left representation is trivially equivalently to the right representation, $R_a = -L_a$. This is the reason why only one-sided representations of Lie algebras are significant in physics.

The notions of isomodules and isobimodules (which were introduced for the first time in Santilli (1979a), and do not appear to have been treated in the mathematical literature, to the author's best knowledge, see the bibliography by Balzer *et al.* (1984)) can then be defined via the one sided and two-sided isotopic liftings, respectively.

According to Santilli (*loc. cit.*), a *Lie-isobimodule* is therefore an isovector space $\hat{V}$ with left and right isocompositions a∗v and v∗a verifying the distributive and scalar laws, and the rules

$$a{*}v = -v{*}a, \tag{D.9a}$$

$$v{*}(a{*}b) = (v{*}a){*}b - (v{*}b){*}a, \tag{D.9b}$$

or, equivalently in terms of isomultiplications

$$\hat{R}_a = -\hat{L}_a, \tag{D.10a}$$

$$\hat{R}_{a*b} = \hat{R}_a{*}\hat{R}_b - \hat{R}_b{*}\hat{R}_a, \tag{D.10b}$$

which characterizes an *isobirepresentation of a Lie-isotopic algebra* $\hat{L}$ as $\mathrm{Hom}^{L}{}_{F}(\hat{V}_R, \hat{V}_L)$. However, the left and right representations are again equivalent because of the property $\hat{R}_a = -\hat{L}_a$. Thus, *only one-sided isorepresentations are needed for the physical applications of Lie-Santilli algebras.*

The notion of isobirepresentations on bimodules becomes *necessary* when passing to the study of the covering Lie-admissible algebras U (Santilli (1979a)). In fact, in this case, the action to the right is no longer equivalent to the action to the left, thus resulting in a much richer structure. In this case a *Lie-admissible bimodule* V has the right and left compositions av and va, such that the attached

composition $a \odot v = av - va$ verifies the conditions

$$a \odot v = - v \odot a, \tag{D.11a}$$

$$v \odot (a \odot b) = (v \odot a) \odot b - (v \odot a) \odot b, \tag{D.11b}$$

which can be equivalently expressed via the right and left multiplications

$$\hat{R}_{ab-ba} + \hat{L}_{ab-ba} = [(\hat{R}_a - \hat{L}_a), (\hat{R}_b - \hat{L}_b)], \tag{D.12}$$

and they characterize a *left and right isobirepresentation of a general Lie-admissible algebra* U as $\mathrm{Hom}^{U}\hat{F}(\hat{V}_R, \hat{V}_L)$.

Santilli (*loc. cit.*) formulated similar structures for commutative Jordan and Jordan-admissible algebras and for other algebras (Sect. 5), but their study is not considered here for brevity.

By recalling Propositions B.1 and C.1 the following property is evident.

PROPOSITION D.1 (Santilli (1991b): An axiomatization of irreversibility from the viewpoint of the representation theory is provided by genorepresentations of Lie-admissible algebras, that is, by modular-isotopic representations with inequivalent axioms to the right and to the left on bimodular vector spaces.

The reader should note the rather remarkable unity of mathematical and physical thought provided by Propositions B.1, C.1 and D.1.

APPENDIX I.E: GENOPARTICLES

Santilli's sequence of representations, isorepresentations and genorepresentations of the preceding appendix were conceived for the characterization of the following sequential physical notions:

A') "PARTICLES", which are characterized by conventional representations of Lie algebras, and consist of the Einsteinian notion of massive point moving in a stable orbit in vacuum under action-at-a-distance, local-potential interactions;

B') "ISOPARTICLES"[39], which are Santilli's more general notion of particle

[39] Santilli's notions of *"isotopic and genotopic mappings"* were recalled in the preceding footnote. The terms *"isoparticles"* and *"genoparticles"* then stand to indicate the *"preservation"* and *"alteration"*, respectively, of the axiomatic structure of the Einsteinian concept of particle.

reviewed in the main text, characterized by isorepresentations of Lie-isotopic algebras, and consist of extended-deformable particles in stable orbit[40] under the most general known, linear and nonlinear, local and nonlocal, potential and nonpotential interactions; and

C') "GENOPARTICLES"[39], which constitute Santilli's most general notion of particle, characterized by genorepresentations of Lie-admissible algebras, and constitute extended-deformable particles under the most general dynamical conditions conceivable at this writing, that is, in nonconservative-unstable orbits while moving within a physical medium under linear and nonlinear, local and nonlocal, and Hamiltonian and nonhamiltonian external forces.

From the content of Appendix D, we can say that

Einstein's notion is a linear, local, one-sided, conventionally modular representation of a Lie algebra.

The Lie-isotopic theory outlined in the main text implies a nontrivial generalization of the preceding notion. In fact,

Santilli's isoparticle is a nonlinear, nonlocal, one-sided, modular-isotopic representation of a Lie-isotopic algebra.

The Lie-admissible formulations outlined in this appendix imply the following further generalization

Santilli's genoparticle is a nonlinear, nonlocal, two-sided, modular-isotopic representation of a Lie-admissible algebra.

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[40] Recall that the Lie-isotopic algebras preserve the antisymmetry of the product of Lie algebras. As such, they characterize conserved quantities which, when representing physical entities, imply stable orbits. As stressed repeatedly by Santilli (1978a, b), (1981a), (1982a), (1991b, d) to prevent physical misrepresentations, the effective treatment of a particle in an unstable (say, decaying) orbit with all algorithms at hand representing physical quantities (e.g., the Hamiltonian H represents the energy of the particle, p represents the linear momentum, etc.), requires the use of the Lie-admissible formulations. These aspects have profound implications for the hadronic structure, which we hope to review in a possible operator sequel of this volume. In fact, they imply that *the hadronic constituents are "isoparticles" only when in stable orbits, otherwise they are "genoparticles"* (Santilli (1989a, b, c, d)). Needless to say, the two formulations are interchangeable, in the sense that Lie-isotopic formulations can also represent stable orbits, but then the algorithms at hand must necessarily lose their physical meaning (e.g., $H = \{ \alpha \exp (\beta t^2) \}$), and this illustrate the insidious possibility of misrepresenting physical results and implications.

It should be mentioned here that we are referring to one of the most complex and, by far, unexplored notions of contemporary mathematics, Santilli's isobirepresentations[41], as expectedly needed to represent some of the most complex physical conditions in the Universe.

On physical grounds, the implications are rather deep. Recall that for Einstein's special relativity a particle is a massive point which, as such, is a perennial and immutable geometric concept. Moreover, the orbits of Einstein's particles are necessarily stable, as trivially requested by the exact character of its rotational subsymmetry.

As shown in Santilli (1988a, c) the Lie-isotopic theory can instead represent the actual shape of the particle considered, as well as all its infinitely possible deformations. Thus, an isoparticle can have an infinite number of different intrinsic characteristics, depending on the infinite number of different interior conditions, and as permitted by the infinite number of isotopes of the Galilei symmetry. However, an isoparticle should always be restricted to a stable orbit

The more general Lie-admissible theory outlined in these appendices implies a further physical generalizations. In fact, besides representing the actual shape of the particle considered and all its possible deformations, Santilli's genoparticle are in unstable orbits, and possess an intrinsically irreversible evolution.

Now, Einstein's notion of particle is unquestionably exact for the arena of its original conception, say, for an electron in an atomic cloud. The inapplicability of the same notion in Santilli's conditions is beyond any credible scientific doubt. In fact, the insistence, say, in the characterization of a proton in the core of a star undergoing gravitational collapse via EinsteinUs notion of particle, would imply that the proton considered freely orbits inside the core of the star with a conserved angular momentum.

The quantum leap in mathematical and physical knowledge offered by Santilli's Lie-isotopic and Lie-admissible formulations is then manifest.

It is evident from the outline of this volume that Santilli's Lie-admissible formulations include, as particular cases, the Lie-isotopic and the conventional Lie formulations. This illustrates the primary mathematical and physical significance of the Lie-admissible formulations over the Lie-Santilli and the conventional Lie formulations.

[41] To this author's best knowledge, only the following isorepresentations of Lie-isotopic algebras have been investigated until now:
a) Isorepresentations of $\hat{O}3)$, Santilli (1989a);
b) isorepresentations of $S\hat{U}(2)$ (Santilli (1989b);
c) fundamental isorepresentations of $S\hat{U}(3)$ (Mignani and Santilli (1991);
no study on the representations of the Lie-admissible algebras has appeared to this writing besides, their proposal (Santilli (1979a)).

Besides constructing, alone and as a theoretical physicist, the entirety of the Lie–isotopic formulations including their physical aPplications outlined in the main text, Santilli was the first to identify the application of the Lie–admissible algebras for the characterization of the algebraic structure of the historical Hamilton's equations with external terms (App. A) at a time, 1967, when only two additional mathematical papers had appeared in rather obscure journals, besides that by Albert (1948).

Santilli subsequently identified the operator counterpart of the above historical equations with external terms, which resulted in his Lie–admissible generalization of Heisenberg's equations (1978b), with consequential proposal to construct a generalization of quantum mechanics via the Lie–admissible generalization of the current Lie structure.

These studies resulted in a new generation of covering mechanics and relativities for interior dynamical conditions fundamentally beyond the technical capabilities of contemporary relativities.

On mathematical grounds, Santilli remains to this day one of the primary contributors in the study of Lie–admissible algebras. In particular, he was the first to formulate the Lie–admissible generalization of enveloping associative algebras, Lie algebras, and Lie groups. Santilli was also the first to introduce the notion of isobimodule and formulate the representation theory of Lie–admissible algebras outlined in the preceding appendix (Santilli (1979).

By keeping in mind the far reaching physical implications of these studies as outlined in this volume, the above events fully justify the listing of Santilli with the year 1967 in the historical chart prepared by the Estonian Academy of Science (reproduced below), among the most illustrious contributors to algebras and physics from Gauss (1820) until this day.

PART II:

LIE–SANTILLI ISOTHEORY

1997

II.1. INTRODUCTION

II.1.1. Limitations of Lie's theory.

As it is well known, *Lie's theory* has permitted outstanding achievements in various disciplines. Nevertheless, in its traditional conception [30] and realization (see, e.g., [15]), Lie's theory is *linear, local-differential and canonical-Hamiltonian*. As such, it possesses clear limitations.

An illustration is provided by the historical distinction introduced by Lagrange [29], Hamilton [14] and other founders of analytic dynamics between the *exterior dynamical problems* in vacuum and the *interior dynamical problems* within physical media. Exterior problems consist of particles which can be effectively approximated as being point-like while moving within the homogeneous and isotropic vacuum under action-at-a-distance interactions (such as a space-ship in a stationary orbit around Earth). The point-like character of particles permits the exact validity of conventional local-differential topologies (e.g., the Zeeman topology in special relativity); the homogeneity and isotropy of space then allow the exact validity of the geometries underlying Lie's theory (such as the symplectic geometry); and the action-at-a-distance interactions assures their representation via a potential with consequential canonical character.

Interior problems consist instead of extended, nonspherical and deformable particles moving within inhomogeneous and anisotropic physical media, with action-at-a-distance as well as contact-resistive interactions (such as a space-ship during re-entry in Earth's atmosphere). In the latter case the forces are of local-differential type (e.g., potential forces acting on the center-of-mass) as well as of nonlocal-integral type (e.g., requiring an integral over the surface of the body), thus rendering inapplicable local-differential topologies; the inhomogeneity and anisotropy of the medium imply the inapplicability of conventional geometries for their quantitative treatment; while contact-resistive interactions violate Helmholtz's conditions for the existence of a potential (the *conditions of variational*

selfadjointness [109]), thus implying the noncanonical character of interior systems.

We can therefore say that Lie's theory in its conventional linear, local and canonical formulation is *exactly valid* for all exterior dynamical problems, while it is *inapplicable* (and not "violated") for the more general interior dynamical problems on topological, geometrical, analytic and other grounds.

II.1.2. The need for a suitable generalization of Lie's theory.

Lie's theory is currently applied to nonlinear, nonlocal and noncanonical systems via their simplifications into more treatable forms, e.g., via the expansion of nonlocal-integral terms into power series in the velocities and then the transformation of the system into a coordinate frame in which it admits a Hamiltonian via the Lie-Koening or the Darboux Theorems [110].

However, however, nonlinear, nonlocal and nonhamiltonian systems cannot be consistently reduced or transformed into linear, local and Hamiltonian ones. An illustration exists in gravitation. The distinction between exterior and interior gravitational problems was in full use in the early part of this century (see, e.g., Schwarzschild's *two* papers, the first celebrated paper [119] on the exterior problem and the second little known paper [120] on the interior problem). The same distinction was also kept in early well written treatises in the field (see, e.g., [4], [38]). The distinction was then progressively abandoned up to the current treatment of all gravitational problems, whether interior or exterior, via the same local-differential Riemannian geometry.

The above trend is based on the belief that interior dynamical problems within physical media can be effectively reduced to a collection of exterior problems in vacuum (e.g., the reduction of a space-ship during re-entry in our atmosphere to its elementary constituents moving in vacuum).

It is important for this study to know that *the exterior and interior problems are inequivalent, and the latter is not exactly reducible to the former.* The inequivalence is established by the fact that *the exterior problem is local-differential and variational selfadjoint* [109], while *the interior problem is nonlocal-integral and variationally nonselfadjoint* [loc. cit.]. This establishes the *inequivalence* on: topological grounds (because the conventional topologies are inapplicable to nonlocal conditions); analytic grounds (because of the lack of a first-order Lagrangian); geometric grounds (because of the inapplicability of conventional geometries to characterize, say, locally varying speeds of light); and other grounds (see monograph [116] for comprehensive studies).

The *irreducibility* of the interior to the exterior problem is established by

the so-called *No-Reduction Theorems* [65] which prohibit the reduction of a macroscopic interior system (such as a satellite during re-entry) *with a monotonically decreasing angular momentum,* to a finite collection of elementary particles each one with *a conserved angular momentum* (see also [116] for comprehensive studies here omitted for brevity).

On geometrical grounds, gravitational collapse and other interior gravitational problems are not composed of ideal points, but instead of a large number of extended and hyperdense particles (such as protons, neutrons and other particles) in conditions of total mutual penetration, as well as of compression in large numbers into small regions of space. This implies the emergence of a structure which is arbitrarily nonlinear (in coordinates and velocities), nonlocal-integral (in various quantities) and non-hamiltonian (variationally nonselfadjoint).

Additional insufficiencies of the current formulation of Lie's theory as well as of its underlying geometries and mechanics exist for the characterization of antimatter. In fact, we possess today effective methods for the characterization of antimatter only at the *operator* level via *charge conjugation.* These methods do not have a counterpart at the *classical* level because charge conjugation is *antiautomorphic* and no corresponding map exists in the classical realization of Lie's theory, as well as in its underlying carriers spaces, geometries and mechanics. There is therefore the need of achieving first a consistent antiautomorphic characterization of antimatter at the *classical-astrophysical* level, and then at the level of its elementary constituents.

Similar occurrences have recently emerged in astrophysics, superconductivity, theoretical biology and other disciplines. These occurrences establish the need for a generalization of the conventional Lie theory which is directly applicable (i.e., applicable without approximation or transformations) to nonlinear, integro-differential and variationally nonselfadjoint equations for the characterization of matter, and then possesses a suitable antiautomorphic map for the effective characterization of antimatter.

II.1.3: Santilli's isotopies and isodualities of Lie's theory.

In a seminal memoir [52] written in 1978 (see also memoir [53] and paper [54] written in the same year) when at Harvard University, the theoretical physicist **Ruggero Maria Santilli** proposed a step-by-step generalization of the conventional formulation of Lie's theory (that is, a generalization of envelopes, algebras, groups, representation theory, etc.) specifically conceived for nonlinear, integro-differential and noncanonical systems. The generalized theory was subsequently studied by Santilli in over one hundred papers (mostly published in

the physical literature), including studies on the structure of the theory and its applications in various fields (see representative papers [52–108]), and then additionally studied in ten monographs [109–118]. The new formulation of Lie's theory which has emerged from these studies is today called the *Lie-Santilli isotopic theory* or *isotheory* for short (see papers [1], [2], [8], [11], [12], [16]–[23], [25], [32], [33], [35]–[37], [40]–[43], [122]–[125], independent monographs [3], [24], [31], [121] and additional references quoted therein).

A main characteristic of the Lie-Santilli isotheory, which distinguishes it from other generalizations, is its *isotopic* nature intended (from the Greek meaning of the word) as the capability of preserving the original Lie axioms. More specifically, Santilli's isotopies [52]–[54] are today referred to *maps of any given linear, local and canonical structure into its most general possible nonlinear, nonlocal and noncanonical forms which are capable of reconstructing linearity, locality and canonicity in certain generalized isospaces and isofields within the fixed inertial coordinates of the observer.*

These properties are remarkable, mathematically and physically, inasmuch as they permit the preservation of the abstract Lie theory and the transition from exterior to interior problems via a more general *realization* of the same theory. We assume the reader is aware of the array of novel problems raised by the above definition of isotopies, such as the representation of *nonhamiltonian* vector fields in the coordinates of the observer *without* Darboux's transformations to an equivalent Hamiltonian form, because the latter, being nonlinear images of the coordinates of the observer, are not realizable in experiments as well as noninertial and, as such, are not usable in practical applications (see the preceding article [100] by Santilli for the solution of this and the other problems connected with the above definition).

It should be indicated that Santilli submitted his isotopic theory in memoir [52] *as a particular* case of a yet more general theory today called *Santilli's Lie-admissible theory* or *Lie-Santilli genotopic theory*, where the term *genotopies* was introduced (from its Greek meaning of "inducing configuration") to denote the characterization of covering Lie-admissible axioms.

In fact, Santilli initiated his research during his Ph. D. studies in theoretical physics at the University of Turin, Italy, by introducing in 1967 [47] a new notion of Lie-admissible algebra with its explicit realization. These early studies in Lie-admissibility were then continued in papers [49]–[53], [55]–[58], and numerous other, as well as in monographs [111], [112].

In essence the *first notion of Lie-admissibility* is due to the American mathematician A. A. Albert (see the historical notes of ref. [52]) and is referred to a nonassociative algebra U with elements a, b, ... and (abstract) product ab whose attached antisymmetric algebra U^-, which is the same vector space as U but

equipped with the product $[a, b]_U = ab - ba$, is Lie. As such, the algebra U does not necessarily contain a Lie algebra in its classification, thus resulting to be inapplicable for the construction of mathematical and physical coverings of Lie's theory.

In fact, Albert was primarily concerned with the requirement that U should contain *Jordan algebras* as particular cases, and conducted his studies with the *quasiassociative algebra* with product

$$(a, b) = \lambda a b + (1 - \lambda) b a, \tag{1.1}$$

where λ is a non-null scalar, which yield a commutative Jordan algebra for $\lambda = \frac{1}{2}$ and ab associative, but which does not admit a Lie algebra under a finite value of λ

The *second notion of Lie-admissibility* was introduced by Santilli in paper [47] as the preceding definition, plus the condition that the algebra U admits Lie algebras in their classification or, equivalently, that the product ab admits as a particular case the Lie product. This definition was presented via the realization of the *flexible Lie-admissible algebras* with product

$$(a, b) = \lambda a b - \mu b a, \tag{1.2}$$

where λ, μ and $\lambda + \mu$ are non-null scalars, under the conditions that

$$[a, b]_U = (a, b) - (b, a) = (\lambda + \mu)(a b - b a), \tag{1.3}$$

is Lie, plus the condition that the product (a, b) admits the Lie product as particular case. The latter conditions are easily met for $\lambda = \mu$ and ab associative.

To the author's best knowledge, paper [47] initiated in 1967 the studies in the so-called "q-deformations" subsequently conducted in the 1980's by a large number of authors with the simpler product

$$(a, b) = a b - q b a, \quad \lambda = 1, \quad \mu = q, \tag{1.4}$$

(although papers in the latter field rarely quote [47]). Santilli also identified in paper [49] of 1969 the first Lie-admissible structure on record of classical dynamics for dissipative systems, thus illustrating the physical need of his "want of a Lie algebra content" [47].

Subsequently, in memoirs [52], [53] of 1978, Santilli introduced the realization of the *general Lie-admissible algebra* $\hat{U}$ with the product

$$(a, b) = a \times R \times b - b \times S \times a, \qquad (1.5)$$

where a×R, R×b, etc. are associative, and R, S, R+S are nonsingular but otherwise arbitrary *operators* with scalars values λ and μ as particular cases. He then discovered that the attached antisymmetric algebras *were not* conventionally Lie with the familiar commutator a×b − b×a, but were instead characterized by the product

$$[a, b]_{\hat{U}} = (a, b) - (b, a) = a \times T \times b - b \times T \times a, \quad T = R + S, \qquad (1.6)$$

which he called *Lie-isotopic* [52], [53]. This resulted in the *third definition of Lie-admissibility*, today called *Albert-Santilli Lie-admissibility*, which refers to a *nonassociative algebra* U *which admit Lie-Santilli isoalgebras both in their attached antisymmetric form* U^- *as well as in their classification.*

Jointly, Santilli identified in the same memoirs a classical [52] and operator [53] realization of the general Lie-admissible algebras, thus establishing the foundations of a structural generalization of Lie-admissible type of analytic and quantum mechanics and of their interconnecting map, of which in this paper we shall merely study the isotopic particular case occurring for $R = S = T = T^{\dagger} \neq 0$.

Albert-Santilli notion of Lie-admissibility can be considered the birth of the Lie-Santilli isotheory, and can be found in Sect. 3 (particularly Sect. 3.7) of ref. [52] and in Sect. 4 (particularly Sect. 4.14) of ref. [53]. In fact, Santilli recognized that the antisymmetric brackets $(a, b)_U$ attached to the *nonassociative* algebra U with product (a, b) = a×R×b − b×S×a can be *identically* rewritten as the antisymmetric brackets attached to an *associative* algebra $\hat{A}$ with product a×$\hat{T}$×b,

$$[a, b]_{\hat{U}} \equiv [a, b]_{\hat{A}}, \qquad (1.7)$$

$$\hat{U}: \ (a, b) = a \times \hat{R} \times b - b \times \hat{S} \times a, \qquad \hat{A}: \ a \times \hat{T} \times b, \ \ \hat{T} = \hat{R} + \hat{S}.$$

The latter identity signaled the transition from studies within the context of *nonassociative algebras* (done by Santilli until 1978), to genuine studies on the generalization of *Lie's theory* (done from 1978 on) via the isotopies of *associative* enveloping algebras and related Lie algebras, Lie groups, representation theory, etc.

In fact, Santilli then discovered that the quantity $\hat{1} = T^{-1}$ is indeed the correct left and right unit of the isotopic envelope $\hat{A}$. *The Lie-Santilli isotheory can therefore be initially conceived as the image of the conventional theory under the lifting of the trivial unit* 1 *of conventional use to a well behaved but otherwise arbitrary unit* $\hat{1}$.

This conception permitted Santilli to identify all main lines of the isotheory already in the original proposal [52], which include: the isotopies of universal enveloping associative algebras (including the isotopies of the fundamental Poincaré-Birkhoff-Witt and Baker-Campbell-Hausdorff theorems); the isotopies of Lie algebras (including the isotopies of the celebrated Lie's first, second and third theorem); the isotopies of Lie transformations groups; and other isotopies.

The original proposal [52] also included the remarkable property of the Lie-Santilli isoalgebra of unifying compact and noncompact simple Lie algebras of the same dimension (see ref. [52], Definition 3.7.2 on the isotopic envelope characterizing *nonisomorphic* Lie algebras with the *same* basis and changing instead T, and the isotopic unification of O(2.1) and O(3) in p. 289). All subsequent developments, including this presentation, have essentially been refinements of these foundations introduced in the original proposal [52], [53].

By the early 1980's Santilli recognized that the available Lie, Lie-isotopic and Lie-admissible formulations could only be applied to matter and not to antimatter for the reasons indicated in Sect. 1.B. He then reinspected his isotopies and in papers [62], [63] (written in 1983 but published in 1985 because of quite unreasonable editorial obstructions by various physics journals reviewed in p. 26 of [62]) he discovered that, once the elementary unit +1 is abandoned in favor of an arbitrary quantity $\hat{1}$, the latter unit admits in a natural way *negative values*. He also discovered that the map $\hat{1} > 0 \rightarrow \hat{1}^d = -\hat{1} < 0$ is *antiautomorphic* precisely as the charge conjugation, and called it *isoduality* in the sense of being a form of duality which necessarily requires the isotopic generalization of the unit.

In the same papers [62], [63] he reformulated the Lie-isotopic theory for negative units $\hat{1}^d$ which is today called *isodual Lie-Santilli isotheory*, and introduced a number of novel notions, such as *isorotational symmetry* $\hat{O}(3)$ and its *isodual* $\hat{O}^d(3)$ which leave invariant the conventional *ellipsoids* with positive semiaxes, and the new *isodual ellipsoids with negative semiaxes*, respectively. He then proved the isomorphism $\hat{O}(3) \approx O(3)$ (and the anti-isomorphism between $\hat{O}^d(3)$ and O(3)), thus disproving the rather popular belief that the rotational symmetry is broken for the ellipsoidical deformations of the sphere (which is correct only under the assumption of realizing Lie's theory in its simplest conceivable form, but incorrect otherwise, as illustrated in Sect. 3.E).

Despite these advances and as admitted in private communications, Santilli abstained from indicating in papers [62], [63] the applicability of the isodual theory for the characterization of antimatter because of its rather deep implications such as a causal motion backward in time, the prediction of antigravity for antiparticles in the field of matter, and others.

After due studies, the above reservation were resolved, and Santilli first

applied his isodual theory for the characterization of antimatter in monographs [113], [114] of 1991. The equivalence between isoduality and charge conjugation was first proved in paper [84] of 1994. Some of the far reaching implications of isoduality were studied in papers [86], [87] of the same year. The first comprehensive treatment of isoduality appeared in the 1994 edition of monograph [116]. The mathematical and physical studies based on isoduality are now rapidly expanding.

The culmination of Santilli's isotopies and isodualities can be seen in the emergence of new notions of space-time and internal symmetries for matter, and their isodual for antimatter which, in turn, culminate in the isotopies and isodualities of the Poincaré symmetry, first proposed by Santilli in paper [59] of 1983 (see paper [79] of 1993 for the latest comprehensive study including its isospinorial covering). The isotopies of the SU(3) symmetry were first studied in paper [34] of 1984 and those of the quark theory in paper [90] of 1995.

The new space-time isosymmetries imply corresponding new classical and quantum mechanics and have far reaching implications, such as: the first exact-numerical representation of the magnetic moment of the deuteron [85] (which has escaped quantum mechanics for three quarters of a century despite all possible relativistic and tensorial corrections); the first exact-numerical representation of the synthesis of the neutron inside new stars from protons and electrons only [95] (which cannot be treated quantitatively by quantum mechanics and quark theories); the consequential prediction of a new source of clean, *subnuclear* energy called "hadronic energy" [88] (all predictive capacities for new energies based on the conventional Poincaré symmetry were exhausted during the first half of this century); and other novel applications, verifications and predictions [116], [118].

In view of the above advances, Santilli received various honors, including the Nomination in 1989 by the Estonia Academy of Sciences among the most illustrious applied mathematicians of all times, jointly with Gauss, Hamilton, Cayley, Lie, Frobenius, Poincaré, Cartan, Riemann, and others, the only member of Italian origin to enter in the list (see the charts of pages 6–7 of ref. [31]). Quite appropriately, the Nomination lists Santilli's first paper [47] on Lie-admissibility written at the University of Turin, Italy, from which everything else follows.

This Part II is solely devoted to the Lie-Santilli isotheory with a few indication of its isodual. In Sect. 2 we shall present the latest formulation of isotopies and isodualities of mathematical methods based on memoir [100]. A comparison with the corresponding formulation of Part I is instructive. The isotopies and isodualities of Lie's theory are presented in Sect. 3.

As an illustration of the capabilities of the Lie-Santilli isotheory, we review in Sect.s 3.D–3.F the "direct universality" of the Poincaré-Santilli isosymmetry, that is, the achievement of the symmetries of all infinitely possible, well behaved, nonlinear, nonlocal and noncanonical generalization of the minkowskian line

element (universality), directly in the coordinates of the observer (direct universality). This universality includes as particular case the symmetry of all possible gravitational models in (3+1)-dimension with consequential unification of the special and general relativities and emergence of a novel quantization of gravity via the *unit* of relativistic quantum mechanics without any need of a Hamiltonian [79], [98]. A number of intriguing open mathematical problems will be identified during the course of our analysis and in the final section.

A comprehensive mathematical presentation of the Lie-Santilli isotheory up to 1992 is available the monograph by Sourlas and Tsagas [121]. A historical perspective is available in the monograph by Lõhmus, Paal and Sorgsepp [31]. The study of continuity properties under isotopies was initiated by Kadeisvili [22]. The first identification of isomanifolds (today called *Tsagas-Sourlas isomanifolds*) was done in ref. [122] which is a topological complement of these algebraic studies.

The author presented the continuous advances in the Lie-Santilli isotheory in reviews [24,25,26]. The present review is a further update over the preceding ones in various details.

In this review we can only quote contributions on the generalization of Lie's theory *based on the broadening of the unit* and we regret our inability at this time to quote the rather numerous contributions on *different* generalization based on the *conventional* unit. The author would be grateful to any colleague who cares to bring to his attention additional relevant literature for quotation in future works.

The author also regret the inability, to avoid a prohibitive length, to indicate the rather intriguing connections existing between the Lie-Santilli isotheory and other generalized formulations, such as the Kac-Moody algebras, superalgebras, quantum algebras, etc., whose study is left to interested mathematicians.

II.2: SANTILLI'S ISOTOPIES AND ISODUALITIES OF CONTEMPORARY MATHEMATICAL METHODS

II.2.1. Introduction.

Santilli has made some of his most momentous advances in pure and applied mathematics by discovering that the formulation of contemporary mathematics with a well defined left and right unit is dependent on the assumption of the simplest conceivable realization of the unit, the scalar number $1 = +1$ or the n-dimensional unit matrix $I = \mathrm{Diag.}(1, 1, 1, \dots)$.

In his memoirs of 1978, Santilli [52], [53] therefore suggested the reconstruction of contemporary mathematics with respect to a quantity $\hat{1}$ of the same dimension of the conventional unit I, but with an unrestricted functional dependence of its elements in time t, coordinates x, their derivatives of arbitrary order, and any needed additional quantity such as local density μ, temperature τ, index of refraction n and, for operator theories, wavefunctions ψ and their derivatives [52], [53], [115],

$$I \quad \rightarrow \quad \hat{1} = \hat{1}(t, x, \dot{x}, \ddot{x}, \psi, \partial\psi, \partial\partial\psi, \mu, \tau, n, \ldots) = \hat{T}^{-1} \qquad (2.1)$$

Jointly, Santilli suggested the lifting of the associative product A×B among generic quantities A, B (e.g., numbers, vector fields, operators, etc.), into the form

$$A \times B \quad \rightarrow \quad A \hat{\times} B = A \times \hat{T} \times B \,. \qquad (2.2)$$

in which case $\hat{1} = \hat{T}^{-1}$ is the correct left and right generalized unit of the theory,

$$\hat{1} \hat{\times} A \equiv A \hat{\times} \hat{1} \equiv A \,, \qquad (2.3)$$

for all possible elements A of the considered set.

For consistency, the entire original mathematics must be reconstructed in such a way to admit the quantity $\hat{1}$ as the correct left and right unit. This implies the reconstruction of numbers and angles, fields and number theory, functional analysis and differential calculus, algebras and geometries, etc.

In reality, rules (2.1)–(2.3) lead Santilli to the discovery of *seven different structural liftings of the contemporary mathematics*, which can be outlined as follows[42]:

1) ***Isomathematics*** [52], which occurs when $\hat{1}$ preserves all the topological characteristics of I, e.g., nowhere-degeneracy, Hermiticity and positive-definiteness.

2) ***Genomathematics*** [52], which occurs when $\hat{1}$ is invertible but non-Hermitean (e.g., a real-valued but non-symmetric matrix);

3) ***Hypermathematics*** [73], which occurs when $\hat{1}$ is a (finite or infinite), ordered *set* of invertible, generally non-Hermitean quantities.

In conventional mathematical and physical formulations, the systems are identified via the sole knowledge of the Hamiltonian H (or of the Lagrangian L). In Santilli's methods, the identification of systems requires the knowledge of *two* different quantities, the Hamiltonian H (or Lagrangian L) and the generalized unit $\hat{1}$.

Isomathematics has resulted to be effective for the representation of closed–

[42] A readable outline can be found in Page 18 of the Web Site M. Battler, M. McBee, and S. Smith **http://home1.gte.net/ibr/**.

isolated systems of particles verifying the usual total conservation laws, with conventional interactions represented with H (or L) plus internal nonlinear, nonlocal-integral and nonpotential-nonhamiltonian interactions represented with $\hat{1}$. Since all known action-at-a-distance interactions are *reversible* (i.e., invariant under time reversal), this first class of systems is globally reversible, namely, their center-of-mass trajectories are reversible from the property of the isounit $\hat{1} = \hat{1}^{\dagger}$ which is then generally assumed to be time-reversal invariant, i.e. $\hat{1}(-t) = \hat{1}(t)$.

Genomathematics applies for the representation of open-nonconservative, nonlocal and non-Hamiltonian systems in *irreversible* conditions. In this case the Hamiltonian H (or Lagrangian L) remain fully reversible, while irreversibility is represented via the axioms of the theory from the property $\hat{1} \neq \hat{1}^{\dagger}$. As we shall see better in Chapter 7, genomathematics therefore represents irreversible systems irrespective of whether $\hat{1}$ is time-dependent or not.

Hypermathematics is significant for quantitative representations of more complex multivalued systems, e.g., of biological type.

Moreover, in 1985 Santilli [62], [63] discovered a new *anti*-isomorphic map of a generic quantity A (again, numbers, vector fields, operators, etc.) into its anti-Hermitean form

$$A \rightarrow A^d = -A^{\dagger}, \qquad (2.4)$$

which he called *isoduality*, and which was relegated by this author in the first edition of this book in 1992 to a possible re-interpretation of the inversions (see the first edition, pages 23, 24).

Momentous advances have been done by Santilli since that time with the constructions of the foundations of yet novel isodual mathematics and their use for a basically novel theory of antimatter.

On physical grounds, Santilli [84,106-108] lamented the dramatic disparity existing in the physics of this century between the treatment of matter and antimatter. In fact, matter is represented at *all* level of current mathematical and physical knowledge from Newtonian mechanics all the way to second quantization, while antimatter is represented only in *second quantization*.

This disparity in treatment implies predictable shortcomings. For instance, Santilli [loc. cit.] proved that *the operator image of the current representation of antimatter is not the correct charge conjugate state, but merely the state of a particle with reversed sign of the charge*. This is evidently a consequence of the use in contemporary physics of *only one channel of quantization*, that for particles.

Deeper shortcomings of the current theory of antimatter were identified by Santilli [105] in the problem of unified gauge theories with the inclusion of gravitation, which we can review only after acquiring the technical background.

To resolve these shortcomings, Santilli therefore constructed a novel theory of antimatter which is applicable *beginning* at the Newtonian level, has its own channel of quantization, and then admits an operator formulation which is equivalent to that provided by charge conjugation [84,106-108].

The guiding principle was the property of charge conjugation of being *anti-automorphic*. He therefore looked for all possible maps which:

1) were also anti-automorphic (or, more generally, anti-isomorphic) like charge conjugation;

2) while charge conjugation is solely applicable at the operator level, the needed map had to be applicable at all classical and quantum level; and

3) the emerging new theory had to represent all available experimental data on antimatter, including the equivalence to charge conjugation at the operator level.

The isodual map (2.4) resulted to verify all the above conditions. In fact, map (2.4) implies, first, the isoduality of the basic unit

$$\hat{1} \rightarrow \hat{1}^d = -\hat{1}^\dagger, \tag{2.5}$$

with consequential isoduality of the product of generic quantities A, B,

$$A \hat{\times} B \rightarrow A^d \hat{\times}^d B^d = (-A^\dagger) \times (-\hat{T}^\dagger) \times (-B^\dagger), \tag{2.6}$$

under which $\hat{1}^d = (\hat{T}^d)^{-1}$ is, again, the correct right and left generalized unit,

$$\hat{1}^d \hat{\times}^d A^d = A^d \hat{\times}^d \hat{1}^d \equiv A^d. \tag{2.7}$$

For consistency the totality of the original mathematical methods must be subjected to the isodual map, thus resulting in still new isodual mathematics, with yet new numbers and angles, new vector and metric spaces, new functional analysis and differential calculus, new algebras and geometries, etc.

Most importantly, the isodual mathematics resulted to be anti-isomorphic to the original mathematics, as desired, thus verifying the crucial condition 1) above. Conditions 2) and 3) will be studied later on.

For readers who are still skeptical on the existing of yet novel mathematics, it is sufficient to note that a mathematics with *negative* unit is conceptually, topologically and geometrically different than a mathematics with *positive* unit, thus warranting separate studies.

These studies on antimatter lead Santilli [100] to the identification of the following *four additional novel mathematics:*

4) ***Isodual mathematics*** [100], which is the isodual image of the *conventional*

mathematics used for the characterization of antiparticles in vacuum;

5) ***Isodual isomathematics*** [100], which is the isodual image of the isomathematics, and it is used for the study of antiparticles in interior conditions with global reversibility;

6) ***Isodual genomathematics*** [100], which is the isodual image of the genomathematics used for the characterization of antiparticles in interior conditions and irreversible global behavior; and

7) ***Isodual hypermathematics*** [100], which is the isodual image of the hypermathematics used for the description of complex multivalued systems systems.

It should be indicated for clarity Santilli's insistence on the fact that *the iso-, geno- and hyper-mathematics do not constitute "new" mathematics because, by conception and construction, they verify exactly the same abstract axioms of the contemporary mathematics with a unit, thus being "new realizations" of existing axioms.* The isodual iso-, geno- and hyper-mathematics verify the abstract axiom of the isodual mathematics and, as such, are anti-isomorphic to the preceding ones.

In turn, this mathematical conception has rather intriguing and far reaching physical implications. As an example, the new realizations of the abstract axioms of quantum mechanics permitted by the iso-, geno- and hyper-mathematics have resulted to be a form of "completion" of quantum mechanics much along the historical teaching of Einstein, Podolsky and Rosen, as indicated in memoir [101] beginning with the title.

What is rather remarkable is that Santilli identified each of the seven new mathematics because of specific physical or biological needs and also constructed the foundations of each of them as necessary pre-requisite for his novel applications.

Needless to say, despite the volume of research conducted to date, the studies are still at their initiation and so much remains to be done. Also, in this monograph we cannot possibly study all the above new mathematics and their applications, and are therefore forced to restrict our attention to more specific goals.

In this Part II we shall solely study the isomathematics and its isodual, with the isodual mathematics being a simple particular case. An outline of the current formulation of the broader genomathematics and hypermathematics cannot be made for brevity (SEE REF. [100,101]).

II.2.2: Classification of isomathematics

The iso- (as well as the geno- and hyper) -mathematics has a rich structure requiring an internal classification for proper study of the various individual aspects.

When generically referred to a formulation with a Hermitean unit, isomathematics was classified by this author [22] into the following classes, today

called *Kadeisvili's Classes*:

Class I (with generalized units that are smooth, bounded, nondegenerate, Hermitean and positive-definite, characterizing the isotopies properly speaking);

Class II (the same as Class I although $\hat{1}$ is negative-definite, characterizing isodualities);

Class III (the union of Class I and II);

Class IV (Class III plus isounits admitting zeros); and

Class V (Class IV plus unrestricted generalized units, e.g., realized via discontinuous functions, distributions, lattices, etc.).

All isotopic structures identified below also admit the same classification which will be omitted for brevity. In this monograph we shall generally study isotopies of Classes I and II, at times treated in a unified way via those of Class III whenever no ambiguity arises.

Santilli's isomathematics of Classes IV and V are vastly unexplored at this writing. This is unfortunate because we know today that, *the zeros of the isounits represent gravitational singularities*, and have other intriguing physical meanings.

II.2.3: Isotopies and isodualities of fields

Santilli's first important contribution to mathematics has been the identification of new numbers and fields with arbitrary units, which he presented for the first time at the meeting on *Differential Geometric Methods in Mathematical Physics* held at the University of Clausthal, Germany, in 1980 (see the latest study [73] and the general presentation in [115]). The basic isotopies are therefore those of fields from which all other isotopies can be derived in a unique and unambiguous way via mere compatibility arguments.

Let $F = F(a,+,\times)$ be a field (hereon assumed to have characteristic zero) with elements $a, b, \ldots$, sum $a + b$, multiplication $a\times b = ab$, additive unit 0, multiplicative unit I, and familiar properties $a + 0 = 0 + a = a$, $a\times I = I\times a = a$, $\forall a \in F$, and others. We have in particular: the field $R(n,+,\times)$ of real numbers n, the field $C(c,+,\times)$ of complex numbers c, and the field $Q(q,+,\times)$ of quaternions q.

Definition 2.1 [73]: *"Santilli's isofield" of Class III* $\hat{F} = \hat{F}(\hat{a},+,\hat{\times})$ *are rings with elements* $\hat{a} = a\times\hat{1}$, *called "isonumbers", where* $a \in F$, *and* $\hat{1} = \hat{T}^{-1}$ *is a Class III element generally outside F, equipped with two operations* $(+, \hat{\times})$, *the "isosum" + which is equivalent to the conventional sum of F and the "new" isoproduct"* $\hat{\times}$

$$\hat{a}\,\hat{\times}\,\hat{b} = \hat{a}\times\hat{T}\times\hat{b}, \quad \hat{1} = \hat{T}^{-1}, \tag{2.8}$$

which is such that $\hat{1} = \hat{T}^{-1}$ *is the left and right unit of* $\hat{F}$,

$$\hat{1} \hat{\times} \hat{a} = \hat{a} \hat{\times} \hat{1} \equiv \hat{a}, \ \forall \ \hat{a} \in \hat{F}, \tag{2.9}$$

in which case (only) $\hat{1}$ *is called "isounit" and* $\hat{T}$ *is called the "isotopic element". Under these assumptions* $\hat{F}$ *is a field, i.e., it satisfies all properties of* F *in their isotopic form for all possible* a, b, c $\in$ F *and* $\hat{a}$, $\hat{b}$, $\hat{c} \in \hat{F}$

1. *The set* $\hat{F}$ *is closed under the isosum,* $\hat{a} + \hat{b} = (a + b)\times\hat{1} \in \hat{F}$;
2. *The isosum is commutative,* $\hat{a} + \hat{b} = \hat{b} + \hat{a}$;
3. *The isosum is associative,* $\hat{a} + (\hat{b} + \hat{c}) = (\hat{a} + \hat{b}) + \hat{c}$;
4. *There is an element* $\hat{0} = 0$ *in* $\hat{F}$ *called 'additive isounit" which is such that* $\hat{a} + \hat{0} = \hat{0} + \hat{a} = \hat{a}$;
5. *For each element* $\hat{a} \in \hat{F}$, *there is an element* $-\hat{a} \in \hat{F}$, *called the "isoopposite" of* $\hat{a}$, *which is such that* $\hat{a} + (-\hat{a}) = 0$;
6. *The set* $\hat{F}$ *is closed under the isoproduct,* $\hat{a}\hat{\times}\hat{b} \in \hat{F}$;
7. *The isoproduct is generally non-isocommutive,* $\hat{a}\hat{\times}\hat{b} \neq \hat{b}\hat{\times}\hat{a}$, *but "isoassociative",* $\hat{a}\hat{\times}(\hat{b}\hat{\times}\hat{c}) = (\hat{a}\hat{\times}\hat{b})\hat{\times}\hat{c}$;
8. *The quantity* $\hat{1}$ *in the factorization* $\hat{a} = a\times\hat{1}$ *is the "multiplicative isounit" of* $\hat{F}$ *as per Eq.s (2.9);*
9. *For each element* $\hat{a} \in \hat{F}$, *there is an element* $\hat{a}^{-\hat{1}} \in \hat{F}$, *called the "isoinverse", which is such that* $\hat{a}\hat{\times}(\hat{a}^{-\hat{1}}) = (\hat{a}^{-\hat{1}})\hat{\times}\hat{a} = \hat{1}$;
10. *The set* $\hat{F}$ *is closed under joint isosum and isoproduct,*

$$\hat{a} \hat{\times} (\hat{b} + \hat{c}) \in \hat{F}, \quad (\hat{a} + \hat{b}) \hat{\times} \hat{c} \in \hat{F}; \tag{2.10}$$

11. *All elements* $\hat{a}$, $\hat{b}$, $\hat{c} \in \hat{F}$ *verify the right and left "isodistributive laws"*

$$\hat{a} \hat{\times} (\hat{b} + \hat{c}) = \hat{a} \hat{\times} \hat{b} + \hat{a} \hat{\times} \hat{c}, \qquad (\hat{a} + \hat{b}) \hat{\times} \hat{c} = \hat{a} \hat{\times} \hat{c} + \hat{b} \hat{\times} \hat{c}. \tag{2.11}$$

When there exists a least positive isointeger $\hat{p}$ *such that the equation* $\hat{p}\hat{\times}\hat{a} = 0$ *admits solution for all elements* $\hat{a} \in \hat{F}$, *then* $\hat{F}$ *is said to have "isocharacteristic* p*". Otherwise,* $\hat{F}$ *is said to have "isocharacteristic zero". Unless otherwise stated, all isofields considered hereon shall be Class III isofields of isocharacteristic zero.*

The above definition contains as a particular case the *isonumbers* (of Class I), $\hat{a}_I = \hat{a} = a\times\hat{1}$, $\hat{1} = \hat{1}^{\dagger} > 0$ with related *isofields* $\hat{F}_I = \hat{F} = \hat{F}(\hat{a},+,\hat{\times})$, and the *isodual isonumbers* (of Class II) $\hat{a}_{II} = \hat{a}^d = a\times\hat{1}^d$, $\hat{1} = -\hat{1} < 0$ and related *isodual isofields* $\hat{F}_{II} = \hat{F}^d = \hat{F}^d(\hat{a}^d,+,\hat{\times}^d)$. Whenever no subscript is indicated, isofields are referred to those of Class I. Definition 2.1 therefore implies:

a) The conventional fields $R(n,+,\times)$ of real numbers n with unit $1 = +1$;

b) the conventional field $C(c,+,\times)$ of complex numbers $c = n_1+i\times n_2$ with unit $1 = +1$;

c) The conventional field $Q(q,+,\times)$ of quaternions q also with unit $1 = +1$;

d) The isodual field $R^d(n^d,+,\times^d)$ of *isodual numbers* $n^d = n\times 1^d$ with isodual unit $1^d = -1$;

e) The isodual field $C^d(c^d,+,\times^d)$ of *isodual complex numbers* $c^d = -c^\dagger = -n_1 + i\times n_2$ with isodual unit $1^d = -1$;

f) The isodual field $Q^d(q^d,+,\times^d)$ of *isodual quaternions* with isodual unit $\hat{1}^d$;

g) The isofield $\hat{R}(\hat{n},+,\hat{\times})$ of *isoreal numbers* $\hat{n} = n\times\hat{1}$ with isounit $\hat{1}$ as in Eq.s (2.1);

h) The isofield $\hat{C}(\hat{c},+,\hat{\times})$ of *isocomplex isonumbers* $\hat{c} = c\times\hat{1}$ with isounit $\hat{1}$;

i) the isofield $\hat{Q}(\hat{q},+,\hat{\times})$ of *isoquaternions* $\hat{q} = q\times\hat{1}$ with isounit $\hat{1}$ (see [73] for the *isooctonions*);

l) The isodual isofield $\hat{R}^d(\hat{n}^d,+,\hat{\times}^d)$ of *isodual isoreal numbers* $\hat{n}^d = n\times\hat{1}^d = -n\times\hat{1}$ with isodual isounit $\hat{1} = -\hat{1}$;

m) The isodual isofield $\hat{C}^d(\hat{c}^d,+,\hat{\times}^d)$ of *isodual isocomplex numbers* $\hat{c}^d = \bar{c}\times\hat{1}^d = -\bar{c}\times\hat{1}$ with isodual isounit $\hat{1}^d$;

n) The isodual isofield $\hat{Q}^d(\hat{q}^d,+,\hat{\times}^d)$ of *isodual isoquaternions* $\hat{q}^d = q^\dagger\times\hat{1}^d = -q^\dagger\times\hat{1}$ with isodual isounit $\hat{1}^d$ (see Ref. [73] for the isodual isooctonions).

The following property can be trivially proved:

Proposition 2.1 [73]: *Isofields $\hat{F}(\hat{a},+,\hat{\times})$ (of Class I) are locally isomorphic to the ordinary fields $F(a,+,\times)$ and the lifting $F \rightarrow \hat{F}$ is therefore an isotopy. Isodual isofields $\hat{F}^d(\hat{a}^d,+,\hat{\times}^d)$ (of Class II) are instead anti-isomorphic to $F(a,+,\times)$.*

Note that, the multiplication for the isonumbers is

$$\hat{a}\,\hat{\times}\,\hat{b} = \hat{a}\times\hat{T}\times\hat{b} = (a\times b)\times\hat{1}, \tag{2.12}$$

while that for isodual isonumbers is

$$\hat{a}^d\,\hat{\times}^d\,\hat{b}^d = \hat{a}^d\times\hat{T}^d\times\hat{b}^d = -(a^\dagger\times b^\dagger)\times\hat{1}. \tag{2.13}$$

It is evident that all operations dependent on the multiplication on F are generalized in a simple yet unique and significant way on $\hat{F}(\hat{a},+,\hat{\times})$ and, separately, on $\hat{F}^d(\hat{a}^d,+,\hat{\times}7d)$, thus yielding isotopies and isodualities of powers, quotients, square roots, etc. We therefore have the following *isopower*, *isosquareroot* and *isoquotient*,

$$\hat{a}^{\hat{n}} = \hat{a} \hat{\times} \hat{a} \hat{\times} \ldots \hat{\times} \hat{a} = (a^n) \times \hat{1}, \tag{2.14a}$$

$$\hat{a}^{\hat{\frac{1}{2}}} = a^{\frac{1}{2}} \times \hat{1}^{\frac{1}{2}}, \quad (\hat{a}^{\hat{\frac{1}{2}}})^{\hat{2}} = (\hat{a}^{\hat{\frac{1}{2}}}) \hat{\times} (\hat{a}^{\hat{\frac{1}{2}}}) = \hat{a}, \tag{2.14b}$$

$$\hat{a} \hat{/} \hat{b} = (\hat{a} / \hat{b}) \times \hat{1} = \hat{c}, \quad \hat{a} = \hat{c} \hat{\times} \hat{b}, \tag{2.14c}$$

with the corresponding *isodual isopower, isodual isosquare root,* and *isodual isoquotient*

$$\hat{a}^{d \hat{n} d} = \hat{a}^d \hat{\times}^d \hat{a}^d \hat{\times}^d \ldots \hat{\times}^d \hat{a}^d = (a^{\dagger n}) \times \hat{1}^d, \tag{2.15a}$$

$$\hat{a}^{d \hat{\frac{1}{2}} d} = \hat{a}^{d \frac{1}{2}} \times \hat{1}^{\frac{1}{2}}, \quad (\hat{a}^{d \hat{\frac{1}{2}} d})^{\hat{2} d} = (\hat{a}^{d \hat{\frac{1}{2}} d}) \hat{\times}^d (\hat{a}^{d \hat{\frac{1}{2}} d}) = \hat{a}^d, \tag{2.15b}$$

$$\hat{a}^d \hat{/}^d \hat{b}^d = (\hat{a}^d / \hat{b}^d) \times \hat{1}^d = \hat{c}^d, \quad \hat{a}^d = \hat{c}^d \hat{\times}^d \hat{b}^d. \tag{2.15c}$$

We have in this way the following novel interpretation that the imaginary unit $i = (-1)^{\frac{1}{2}}$ is the ordinary square of the isounit unit $1^d = -1$, $i = (1^d)^{\frac{1}{2}}$ [106].

Note that isounits $\hat{1}$ and, independently, isodual isounits $\hat{1}^d$, satisfy all axiomatic condition for a unit,

$$\hat{1}^{\hat{n}} = \hat{1} \hat{\times} \hat{1} \hat{\times} \ldots \hat{\times} \hat{1} \text{ (n-times)} = \hat{1}, \quad \hat{1}^{\hat{\frac{1}{2}}} = \hat{1}, \quad \hat{1} \hat{/} \hat{1} = \hat{1}, \text{ etc.} \tag{2.16a}$$

$$\hat{1}^{d \hat{n} d} = \hat{1}^d \hat{\times}^d \hat{1}^d \hat{\times}^d \ldots \hat{\times}^d \hat{1}^d \text{ (n-times)} = \hat{1}^d, \quad \hat{1}^{d \hat{\frac{1}{2}} d} = \hat{1}^d, \quad \hat{1}^d \hat{/}^d \hat{1}^d = \hat{1}^d, \text{ etc.} \tag{2.16b}$$

The *isonorm* and *isodual isonorm* are defined respectfully by

$$\hat{|} \hat{a} \hat{|} = |a| \times \hat{1}, \quad \hat{|} \hat{a}^d \hat{|}^d = |a| \times \hat{1}^d, \tag{2.17}$$

where $|a|$ is the conventional norm. As a result, *isonorms* are *positive-definite while isodual isonorms* are *negative-definite.*

This implies that all quantities which are conventionally positive, become negative-definite under isodualities. When represented with the isodual theory, *antimatter has the opposite charge of matter, as well as negative mass, negative energy, negative (magnitude of the) angular momentum, negative dimensions, negative entropy, etc., and, inevitably, moves backward in time (negative time).*

Note that all conventional objections against negative masses and time are *inapplicable* to the isodual theory, trivially, because they are tacitly referred to conventional positive units. In fact, *negative* characteristics referred to *negative* units are fully equivalent on all grounds (including causality), although anti-isomorphic, to *positive* characteristics referred to *positive* units.

While studying the isodual numbers, Santilli [84,106=108] discovered a new invariance which has resulted to have a fundamental physical relevance, from Newtonian mechanics all the way to unified gauge theories inclusive of gravitation, and which can be introduced as follows:

Definition 2.2 [loc. cit.]: *A generic quantity A (i.e., a number, a matrix, an operator, etc.) is said to be "isoselfdual" when it is invariant under isoduality* (2.4),

$$A \equiv A^d = -A^\dagger. \tag{2.18}$$

As we shall see later on, Santilli discovered the above new invariance in the conventional Dirac equations because *Dirac's gamma matrices are isoselfdual*, $\gamma_\mu \equiv \gamma_\mu{}^d$. This so simple a mathematical property has far reaching physical implications. As we shall study later on, it implies that, contrary to popular beliefs throughout this century, *the conventional Poincaré symmetry "cannot" be the true, axiomatically correct symmetry of Dirac's equations because it is not isoselfdual.*

At this stage we merely present as an illustration the fact that the conventional imaginary unit i *is isoselfdual,*

$$i^d = -\bar{i} \equiv i. \tag{2.19}$$

The latter occurrence explains better the correct isodual conjugation for complex numbers,

$$c^d = (n_1 + i \times n_2)^d = n_1{}^d +^d i^d \times^d n_2{}^d = -n_1 + i \times n_2. \tag{2.20}$$

Note that the isotopies and isodualities are restricted to the product, as indicated by the preservation of the symbol + and the change of the symbol × in $\hat{F}(\hat{a},+,\hat{\times})$ and $\hat{F}^d(\hat{a}^d,+,\hat{\times}^d)$. This is due to the fact that the lifting of a field into the form $\hat{F}(\hat{a},\hat{+},\hat{\times})$ inclusive of the lifting of the sum, such as

$$a + b \ \rightarrow \ a \hat{+} b = a + K + b, \tag{2.21}$$

with corresponding lifting of the additive unit

$$0 \rightarrow \hat{0} = -K, \quad K > 0, \quad K \in F, \tag{2.22}$$

generally implies the loss of the original axioms, such as the loss of closure under the distributive law. Therefore, the lifting of the sum is not an isotopy [73]. Moreover,

series which are convergent on $F(a,+,\times)$ as well as on $\hat{F}(\hat{a},\hat{+},\hat{\times})$, such as the exponentiation $1 + a/1! + a^2/2! + ... = e^a$, become divergent under the liftings of the sum

$$1 \hat{+} a/1! \hat{+} a^2/2! \hat{+} ... = \infty . \tag{2.23}$$

For this reason the isotopies and isodualities of the sum are *not* used in applications (see Ref. [129] for their mathematical study).

Despite its simplicity, the lifting $F \rightarrow \hat{F}$ has significant implications in number theory itself. For instance, real numbers which are not conventionally prime under the *tacit* assumption of the unit 1 can become prime under a different unit. In fact, the number 4 is prime under the isounit $\hat{1} = 3$.

This illustrates that most of the properties and theorems of the contemporary number theory are dependent on the assumed unit and, as such, admit intriguing isotopies yielding the *isonumber theory* [115]

As an example of application of Santilli's isonumbers independent from those studied in this volume, the isotopies permit the conception of a new generation of cryptograms called *isocryptograms* [115], which are expected to be more difficult to break then conventional ones because of the availability of an *infinite number of different units* which are not admitted by conventional cryptograms via the conventional number theory.

To prevent misrepresentations of subsequent sections which often remains undetected, the reader is recommended to get acquainted with the new numbers prior to the studies of subsequent aspects. For instance, the traditional statement two × two = four" has no correct mathematical meaning in Santilli's theories because it lack the identification of the assumed unit as well as of the assumed product. And in fact, under the generalized unit $\hat{1} = 3$, "two × two = twelve", the understanding is that, in general, "two × two = integro-differential quantity".

Similarly, the reader should keep in mind that the ordinary negative numbers $-n \in R$ have no connection with Santilli's isodual numbers $n^d = -n \in R^d$. This is evidently due to the fact that the isodual unit $I^d = -1$ *is not* the unit of negative numbers in R because $I^d \times (-n) = +n \neq -n$.

In closing this section it may be useful to visualize with specific examples already in these introductory aspects the types of isounits used in applications. A most important property of the isotopies $F(a,+,\times) \rightarrow \hat{F}(\hat{a},\hat{+},\hat{\times})$ is that the isounit $\hat{1}$ can be *outside* the original field. This implies in particular that, starting from conventional real numbers $n \in R$, the isoreal isonumbers $\hat{n} \in \hat{R}$can be *matrices* whose elements are nonlinear integro-differential functions.

The latter degrees of freedom in the selection of the isounit is of paramount importance in practical applications. In fact, while a Hamiltonian H or a Lagrangian L

represents all possible action-at-a-distance interactions, the isounits can represents all characteristics which are outside the representational capabilities of H or L.

One of the simplest possible example is *the use of the isounit for the representation of extended, nonspherical and deformable shapes*, which is evidently outside any realistic possibility of being represented with H or L. In fact, extended, nonspherical and deformable shapes of ellipsoidical type are represented with the diagonal isounit

$$\hat{1} = \mathrm{diag.}\,(\, n_1^2\, , n_2^2, n_3^2\,)\,, \tag{2.24}$$

where the quantities n_k^2 are real valued and positive-definite functions of local quantities, such as the intensity of external fields, the local pressure, etc., while more general shapes are represented with nondiagonal realizations.

The representation of extended-deformable particles with local-differential, nonpotential forces is done with isounits of the type

$$\hat{1} = \mathrm{diag.}\,(\, n_1^2\, , n_2^2, n_3^2\,) \times e^{\,f(t,\, x,\, \dot{x},\, ..)}\,. \tag{2.25}$$

where the diagonal matrix represents the shape of the particle and the exponential function represents the nonpotential forces (see later on Sect. 2.8 for examples).

An illustration of nonlocal-integral forces (i.e., forces depending on a surface or volume integral) is given by the *Animalu isounit*,

$$\hat{1} = (\, e^{\int d^3x\, \phi^\dagger_\uparrow(x)\, \phi_\downarrow(x)}\,) \times \mathrm{diag.}\,(1,\, 1,\, 1)\,, \tag{2.26}$$

which permits a quantitative representation of the attraction among the two identical electrons of the Cooper pair in superconductivity in a way conform with experimental evidence [95], where $\phi_\uparrow$ and $\phi_\downarrow$ are the wavefunctions of the two electrons with related spin orientation $\uparrow$ and $\downarrow$.

In general, the isounit used in application is a diagonal matrix with the dimension of the carrier space (two-, three- and four-dimension for problems in the plane, space and space-time, respectively) whose elements have a generally linear and nonlinear, local-differential as well as nonlocal integral dependence on local physical quantities.

In summary, Santilli main discovery in Ref. [73] is that the abstract axioms of a field do not require that the basic unit must necessarily be the trivial number +1, because it can also be an arbitrary nonsingular and Hermitean quantity, yielding the isofields and isodual isofields.

As outlined in Chapter 7, Santilli moreover showed that the same axioms of a

field do not necessarily need a Hermitean [73] or a one–valued units [100]. This lead to the the additional discovery of the *genonumbers* with non–Hermitean units, and the *hypernumbers with nonhermitean multivalued units.*

The reader is suggested to meditate a moment on the following aspects implied by the presentation of this section, that the entire contemporary mathematical knowledge is not apparently applicable for quantitative studies of antimatter, or that the same mathematical knowledge is based on the simplest possible unit +1 which has essentially remained unchanged since its inception dating back to biblical times.

One can therefore see the horizon of new possibilities permitted by the generalization of such a fundamental notion.

II.2.4: Isotopies and isodualities of metric and pseudo-metric spaces

Santilli's second important contribution to mathematics has been the identification of a structural generalization of the conventional notions of vector and metric (or pseudo–metric) spaces, first presented in paper [59] of 1983 (see monographs [115] for detailed treatments). In this section we shall review the main lines of the isotopies of metric (or pseudo–metric) spaces.

Recall that conventional metric spaces are defined over a field. It is then easy to see that the lifting of a field requires, for necessary consistency, a corresponding lifting of metric spaces. In turn, such a lifting is at the foundation of the representation of nonlinear, nonlocal–integral and nonhamiltonian systems.

Let $S = S(x,g,R)$ be an N–dimensional metric or pseudo–metric space, with local chart $x = \{x^k\}$, $k = 1, 2, ..., n$, n–dimensional, nowhere singular, real–valued and symmetric metric $g = g(x, ...)$ and invariant separation between two points $x, y \in S$ over the reals

$$(x - y)^2 = (x^i - y^i)\, g_{ij}\, (x^j - y^j) \in R(n,+,\times), \tag{2.27}$$

where the convention on the sum of repeated indices is assumed hereon.

Definition 2.3 [59, 115]: *"Santilli's isospaces" of Class III* $\hat{S} = \hat{S}(\hat{x},\hat{g},\hat{R})$ *are N–dimensional metric or pseudo–metric spaces defined over an isoreal isofield of Class III* $\hat{R}(\hat{n},+,\hat{\times})$ *with* a common *N×N–dimensional isounit* $\hat{1}$ of the same class, *equipped with the "isometric"*

$$\hat{G} = \hat{g} \times \hat{1} = (\hat{g}_{ij}) \times \hat{1} = (\hat{T} \times g) \times \hat{1}, \qquad \hat{1} = \hat{T}^{-1}, \tag{2.28}$$

local chart in contravariant and covariant forms

$$\hat{x} = \{\hat{x}^k\} = \{x^k \times \hat{1}\}, \quad \hat{x}_k = \hat{\delta}_{ki}\,\hat{x}^i = \hat{T}_k{}^r\,\delta_{ri}\,x^i \times \hat{1}, \quad x^k, x_k \in E; \tag{2.29}$$

and "isoseparation" among two points $\hat{x}, \hat{y} \in \hat{S}$ *on the isoreals*

$$(\hat{x} - \hat{y})^{\hat{2}} = (\hat{x} - \hat{y})^i \hat{\times} \hat{G}_{ij} \hat{\times} (\hat{x}^j - \hat{y}^j) =$$

$$= [(x - y)^i \times \hat{g}_{ij} \times (x - y)^j] \times \hat{1} \in \hat{R}. \tag{2.30}$$

Note that the isoseparation, for consistency, must be an element of the isofield, that is, must have the structure of a number n multiplied by the isounit $\hat{1}$. Similarly, on rigorous mathematical grounds, the "isometric" must be expressed by an "isomatrix", namely, its elements must also be isoscalar, thus having the structure $\hat{n} = n{\times}\hat{1}$. These isoscalar characters are expressed by the isomultiplication

$$\hat{x}^2 = \hat{x}^k \hat{\times} \hat{x}_k = (x^k \times \hat{1}) \times \hat{T} \times (x_k \times \hat{1}) = (x^k \times x_k) \times \hat{1} = n \times \hat{1}. \tag{2.31}$$

But the contraction over the repeated index k is in isospace,. We recover in this way the isoseparation of Def. 2.3,

$$\hat{x}^2 = (x^k \times x_k) \times \hat{1} = (x^i \times \hat{g}_{ij} \times x^j) \times \hat{1}. \tag{2.32}$$

Because of the above occurrences, whenever no confusion arises, isospaces can be practically treated via the conventional coordinates x^k rather than the isotopic ones $\hat{x}^k = x^k{\times}\hat{1}$, and with the isometric $\hat{g}$ rather than $G = \hat{g}{\times}1$, with the understanding that the mathematically correct formulation is that in terms of the isocoordinates $\hat{x} = x{\times}\hat{1}$ and isometrics $\hat{G} = \hat{g}{\times}\hat{1}$.

Under the above understandings, Definition 2.3 includes:

a) The *ordinary spaces* $S(x,g,R)$ over R with unit $1 = I = \mathrm{Diag.}(1, 1, ...)$;

b) The *isodual spaces* $S^d(x^d,g^d,R^d)$ with *isodual coordinates* $x^d = -x$ and *isodual metric* $g^d = -g$ over the isodual field R^d with isodual unit $I^d = \mathrm{diag.}(-1, -1, -1)$;

c) The *isospaces* of Class I $\hat{S}_I = \hat{A} = \hat{S}(\hat{x},\hat{g},\hat{R})$ with *isocoordinates* $\hat{x} = x{\times}\hat{1}$ and isometric $\hat{g} = \hat{T}{\times}g$ over the isofield of isoreal numbers $\hat{R}$ with common N×N-dimensional isounit $\hat{1}$; and

d) The *isodual isofields* $\hat{S}_{II} = \hat{S}^d = \hat{S}^d(\hat{x}^d,\hat{g}^d,\hat{R}^d)$ with *isodual isocoordinates* $\hat{x}^d = -\hat{x}$ and *isodual isometric* $\hat{g}^d = -\hat{g}$ over the isodual isofield $\hat{R}^d$.

Each of the above four classes can then be referred to Euclidean, Minkowskian, Riemannian, Finslerian and any other metric or pseudo-metric space of

the contemporary literature.

The first important property of the lifting $S(x,g,R) \rightarrow \hat{S}(\hat{x},\hat{g},\hat{R})$ is that the joint liftings $1 \rightarrow \hat{1} = \hat{T}^{-1}$ and $g \rightarrow \hat{g} = \hat{T}\times g$ preserves all original geometric axioms.

Proposition 2.2 [59]: *Isospaces* $\hat{S}(\hat{x},\hat{g},\hat{R})$ *(isodual isospaces* $\hat{S}^d(\hat{x}^d,\hat{g}^d,\hat{R}^d)$ *are locally isomorphic (anti-isomorphic) to the original spaces* $S(x,g,R)$ (S^d,x^d,g^d,R^d)*).*

The physical implications of the above simple geometric property are far reaching, as we shall see. As an illustration, recall that all conventional spaces are exactly valid for exterior dynamical problems in vacuum and do not have the functional dependence necessary for an effective representation of interior dynamical problems, such as an arbitrary nonlinearity in the *velocities*, *nonlocal-integral effects*, etc.

Santilli [59,115,116] therefore achieved the capability of quantitative treatment of interior dynamical problems via conventional geometric axioms, thus achieving a remarkable geometric unification of exterior and interior problems.

For instance, the Riemannian geometry possesses a metric g(x) with the sole dependence on the local coordinates and a limited capability to incorporate velocity effects from its affine connections. Santilli showed that the isotopies permit the enlargement of the Riemannian metric to an arbitrary functional dependence

$$g(x) \rightarrow \hat{g} = \hat{T}\times g = \hat{g}(t, x, \dot{x}, \ddot{x}, \psi, \partial\psi, \partial\partial\psi, \mu, \tau, n, \dots), \qquad (2.33)$$

under the sole condition that the isotopic element is of Class I.

The issues immediately raised by the above results is then: why use the conventional Riemannian geometry for interior gravitational problems with a sole functional dependence of the metric on the coordinates when the covering Riemann-Santilli isogeometry is characterized by the same axioms, yet admits an unrestricted functional dependence of the metric for more realistic representations of interior problems?

Another illustration of the implications of Proposition 2.2 is the fact that the isotopies $\hat{\eta} = \hat{T}\times\eta$ of the Minkowski metric $\eta = \text{diag.}(1, 1, 1, -1)$ includes all possible *Riemannian* metrics $g(x) \equiv \hat{\eta}(x) = \hat{T}(x)\times\eta$ under the *Minkowskian* axioms. This permitted Santilli the achievement of a classical and operator *geometric unification of the special and general relativities with the axioms of the special*, with far reaching implications for quantum gravity, unified gauge theories, etc.

Another important property of isospaces is that *they imply the alteration (called "mutation"* [53]) *of the basic units of the original space.*

Consider the Euclidean space $E(r,\delta,R)$ with coordinates $r = \{x, y, z\}$ and metric δ

= diag. (1, 1, 1) over the reals R = R(n,+,×). Its basic geometric and algebraic unit (that is, the unit of the space and of its group of isometries) is the quantity I = diag. (1, 1, 1) which represents in a dimensionless form the units of the Cartesian axes, e.g., +1 cm, +1 cm, +1 cm. By recalling that the isounits of Class I can always be diagonalized, the isotopies then imply the lifting

$$I = \text{diag.}(+1\text{ cm}, +1\text{ cm}, +1\text{ cm}) \rightarrow \hat{1} = \text{diag.}(+n_1^2\text{ cm}, +n_2^2\text{ cm}, +n_3^2\text{ cm}), \quad (2.34)$$

namely, not only the value of the unit in each Cartesian axis is changed, but different Cartesian axes have generally different units.

As a result, isospaces imply simple, yet unique and nontrivial generalizations of conventional notions, such as that of the sphere (see next chapter).

Moreover, for consistency, isospaces must have the same isounit of the underlying base field. This leads to the following structure

$$\text{Isoinvariant} = [\text{Length}]^2 \times [\text{Unit}]^2. \quad (2.35)$$

As we shall see, the above occurrences have additional rather profound geometrical and physical implications, including new symmetries expressing the degrees of freedom of the unit, new predictions, a new form of locomotion called "geometric propulsion" and others.

Note that conventional spaces have a unit I = diag. (1, 1, 1, ...) which is *different* than the unit I = +1 of the base field. The same conventional fields can however be trivially reformulated for the unit I = diag. (1, 1, ...) of the space, in which case (only) they are admitted as particular case of the isospaces and verify structure (2.29) of the basic invariant. Such a redefinition is hereon assumed.

The isodual spaces and isospaces have even more intriguing characteristics and, consequential implications. In fact, The fundamentals mathematical and physical quantity of the *isodual Euclidean space* $E^d(r^d, \delta^d, R^d)$ is the isodual unit I^d = diag. (−1 cm, −1 cm, −1 cm), namely *the space* E^d *is defined over negative-definite Cartesian units*. The isotopies of E^d then lead to the lifting

$$I^d = \text{diag.}(-1\text{ cm}, -1\text{ cm}, -1\text{ cm}) \rightarrow \hat{1} = \text{diag.}(-n_1^2\text{ cm}, -n_2^2\text{ cm}, -n_3^2\text{ cm}), \quad (2.36)$$

namely, the space is defined over units which are not only arbitrary and different for different axes, but also negative-definite.

Yet another important difference between spaces and isospaces is that the former admit only one interpretation, the conventional one over the reals, while the latter admits *two interpretations*, the first over the isoreals (i.e., computed with respect to the isounit $\hat{1}$), and the second when projected in conventional spaces (i.e.,

computed with respect to the conventional unit).

This simple geometric occurrence is at the foundation of Santilli's unification of the Minkowskian and Riemannian geometries studied in the next chapter. As indicated earlier, the isotopies of the Minkowskian space with isometric $\hat{\eta} = \hat{T}\times\eta$ admit as a particular case all possible Riemannian metrics $\hat{\eta} \equiv T(x)\times\eta = g(x)$. The isominkowskian space admits two interpretation, the first over the isoreals, in which case the *Minkowskian* axioms occur, and the second when projected over conventional fields, in which case the *Riemannian* axioms are recovered in their entirety.

The projection of isospaces on conventional spaces can be easily computed for diagonal metrics and isotopic elements via the use of a new conventional space $S(\bar{x},g,R)$ over the reals with coordinates

$$\bar{x}_k = x_k \times \hat{T}_{kk}^{\,1/2}, \tag{2.37}$$

under which we have the identity

$$x^i \times \hat{g}_{ij} \times x^j \equiv \bar{x}^i \times g_{ij} \times \bar{x}^j, \tag{2.38}$$

As we shall see, the above simple rule has considerable pragmatic value in applications.

In order to prevent possible misrepresentations, the reader is suggested to meditate a moment on the above properties prior to initiating the study of additional aspects. For instance, the study of the geodesics of isospaces via conventional units leads to a host of inconsistencies which generally remained undetected by the non-initiated reader.

II.2.5: Isotopies and isodualities of continuity, manifolds and topology

The notion of *isocontinuity of Class I* on an isospace was first studied by Kadeisvili [22] and resulted to be easily reducible to that of conventional continuity. In fact, an *isofunction* has the structure $\hat{f}(\hat{x}) = f(x)\times\hat{1}$. Then, the *isomodulus* $\hat{|}\hat{f}(\hat{x})\hat{|}$ of an isofunction $\hat{f}(\hat{x})$ on isospace $\hat{S}(\hat{x},\hat{g},R)$ over the isofield $\hat{R}(\hat{n},+,\hat{\times})$ is given by the conventional modulus $|\hat{f}(\hat{x})|$ multiplied by the a well behaved isounit $\hat{1}$,

$$\hat{|}\,\hat{f}(\hat{x})\,\hat{|} = |f(x)|\times\hat{1}. \tag{2.39}$$

As an illustration, an infinite sequence $\hat{f}_1, \hat{f}_2, \ldots$ is said to be *strongly isoconvergent*

to $\hat{1}$ when

$$\mathrm{Lim}_{k \to \infty} \uparrow \hat{1}_k - \hat{1} \uparrow = 0, \tag{2.40}$$

while the *isocauchy condition* can then be expressed by

$$\uparrow \hat{1}_m - \hat{1}_n \uparrow < \hat{\delta} = \delta \times \hat{1}, \tag{2.41}$$

where δ is real and m and n are greater than a suitably chosen $N(\delta)$. The isotopies of other notions of continuity, limits, series, etc. can be easily constructed [26]. Note that functions which are conventionally continuous are also isocontinuous. Similarly, a series which is strongly convergent is also strongly isoconvergent.

However, a *series which is strongly isoconvergent is not necessarily strongly convergent* (ref. [115], p. 271). As a result, a *series which is conventionally divergent can be turned into* a *convergent form under* a *suitable isotopy*. This mathematically trivial property has rather important applications, e.g., for the reconstruction of convergence at the isotopic level.

The *isodual isocontinuity of Class II* is a simple isodual image of the preceding notion and its explicit form is left to the interested reader.

The notion of an N-dimensional *isomanifold* and *isotopology of Class I* were first studied by Tsagas and Sourlas [122]. These authors also introduced a *conventional* topology on an isomanifold. The latter was lifted into an *isotopology* by Santilli [100].

All isounits of Class I can always be diagonalized into the form

$$\hat{1} = \mathrm{diag.}\,(n_1^2, n_2^2, ..., n_n^2), \quad n_k(t, x, ...) > 0, \quad k = 1, 2, ..., N, \tag{2.42}$$

Consider then n isoreal isofields $\hat{R}_k(\hat{n},\hat{+},\hat{\times})$ each characterized by the isounit $\hat{1}_k = n_k^2$ with (ordered) Cartesian product

$$\hat{R}^N = \hat{R}_1 \times \hat{R}_2 \times ... \times \hat{R}_N. \tag{2.43}$$

Since $\hat{R}_k \sim R$, it is evident that $\hat{R}^{\hat{N}} \sim R^N$, where R^N is the Cartesian product of N conventional fields $R(n,+,\times)$. But the total unit of $\hat{R}^N$ is expression (2.33). Therefore, one can introduce a topology on $\hat{R}^N$ via the simple isotopy of the conventional topology on R^N,

$$\hat{\tau} = \{\emptyset, \hat{R}^N, \hat{K}_i\}, \tag{2.44}$$

where $\hat{K}_1$ represents the subset of $\hat{R}^n$ defined by

$$\hat{K}_1 = \{ \hat{P} = (\hat{a}_1, \hat{a}_2, ..., \hat{a}_n) ? \hat{n}_1 < \hat{a}_1, \hat{a}_2, ..., \hat{a}_n < \hat{m}_1, \hat{n}_1, \hat{m}_1, a_k \in \hat{R} \} . \qquad (2.45)$$

As one can see, the above topology coincides everywhere with the conventional Euclidean topology τ *of* R^n *except at the isounit* $\hat{1}$. In particular, $\hat{\tau}$ is everywhere local-differential, except at $\hat{1}$ which can incorporate integral terms. The above structure is called the *Tsagas-Sourlas-Santilli isotopology* or *integro-differential topology*.

Definition 2.4 [122,100]: *A "topological isospace" of Class I* $\hat{\tau}(\hat{R}^n)$ *is the isospace of Class I* $\hat{R}^n$ *equipped with the isotopology* $\hat{\tau}$. *A "Cartesian isomanifold" of the same class* $\hat{M}(\hat{R}^N)$ *is the isotopological isospace* $\hat{\tau}(\hat{R}^n)$ *equipped with a vector structure, an affine structure and the mapping*

$$\hat{f}:\ \hat{R}^n \rightarrow \hat{R}^n,\quad \hat{f}:\ \hat{a} \rightarrow \hat{f}(\hat{a}) = \hat{a},\ \ \forall\ \hat{a} \in \hat{R}. \qquad (2.46)$$

An "isoeuclidean isomanifold" of Class I $\hat{M}(\hat{E}(\hat{x},\hat{\delta},\hat{R}))$ *occurs when the* N*-dimensional isospace* $\hat{E}$ *is realized as the Cartesian product*

$$\hat{E}(x,\hat{\delta},\hat{R}) \approx \hat{R}_1 \times \hat{R}_2 \times \ \dots\ \times \hat{R}_n\ , \qquad (2.47)$$

and equipped with the isotopology $\hat{\tau}$ *with isounit* (2.34).

For all additional aspects of isomanifolds and related topological properties we refer the interested reader to Tsagas and Sourlas [122] and to Santilli [100]. The extension of the results to n *isodual isotopology* is trivial and will be assumed herein.

II.2.6. Isotopies and isodualities of functional analysis

The data elaboration under isotopies require a step-by-step lifting of all aspects of functional analysis into a new discipline called by Kadeisvili [22] *isofunctional isoanalysis*. This includes the isotopies of conventional and special functions, distributions and transforms.

For instance, the conventional Dirac delta distribution has no meaning under isotopy, mathematically, because of the loss of applicability of the conventional exponentiation and, physically, because particles are no longer point-like. The *isodirac distribution* [23] is the reconstruction of the conventional distribution for an unrestricted unit permitting a direct treatment of the extended character of particles. The Fourier transform, Legendre polynomials, etc., also admit simple yet unique and unambiguous isotopies with important applications in various disciplines.

Regrettably, we are unable to review the isofunctional isoanalysis in details

and are forced to limit ourselves to a review of the following basic notions (see monograph [115] for detailed presentation as of 1995).

Recall that in the transition from the two-dimensional Euclidean to the Riemannian geometry there is the loss of the trigonometric and hyperbolic functions due to curvature. Under isotopies the situation is different. In fact, we can represent all possible two-dimensional Riemannian metrics in an isoeuclidean space which, as such, satisfies the axioms of flatness in isospace (see next chapter), thus permitting the reconstruction of trigonometric and hyperbolic functions for all possible Riemannian metrics [115].

To outline the latter reconstruction, consider the Three-dimensional *isoeuclidean isospace* $\hat{E} = \hat{E}(\hat{r},\hat{\delta},\hat{R})$

$$\hat{E} = \hat{E}(\hat{r},\hat{\delta},\hat{R}): \quad \hat{r} = \{ r^k \times \hat{1} \}, \quad \hat{\delta} = \hat{T} \times \delta, \; \delta = \text{Diag.}(1, 1, 1),$$

$$\hat{1} = \text{Diag.}(n_1^2, n_2^2, n_3^2). \tag{2.48}$$

An *isoline* on $\hat{E}$ over $\hat{R}$ is the conventional topological notion although referred to *isopoints* with values $\hat{r} = r \times \hat{1}$ on an isofield $\hat{R}$. An *isostraight line* in the isoeuclidean $(\hat{x}, \hat{y})$–isoplane has the form

$$\hat{a} \hat{\times} \hat{x} + \hat{b} \times \hat{y} + \hat{c} = 0, \quad \hat{x}, \hat{y} \in \hat{E}, \quad \hat{a}, \hat{b}, \hat{c} \in \hat{R}, \tag{2.49}$$

although, its projection into E over R is given, in general, by the *curve*

$$[\, a \times y / n_1(r, ...) + b \times y / n_2(r, ...) + c \,] \times \hat{1} = 0 \tag{2.50}$$

Intersecting isostraight isolines then permit a unique and consistent definition of *isoangles* $\hat{\alpha}$ which is impossible in the Riemannian treatment of gravity.

The study of the isoeuclidean geometry has established that *conventional numerical value of angles are preserved under isotopies*, i.e., isotopies map parallel (perpendicular) straight lines into isoparallel (isoperpendicular) isostraight isolines.

The projections of the isoangles $\hat{\phi}$ on the $(\hat{x}, \hat{y})$–isoplane and the angle $\hat{\theta}$ with respect to the $\hat{z}$–axis into E over R assume the forms

$$\hat{\phi} = \phi \times \hat{1}_\phi, \quad \hat{\theta} = \theta \times \hat{1}_\theta. \tag{2.51a}$$

$$\hat{1}_\phi = 1 / n_1 \times n_2 \quad \hat{T}_\phi = n_1 \times n_2, \quad \hat{1}_\theta = 1 / n_3, \quad \hat{T}_\theta = n_3. \tag{2.51b}$$

The *isotrigonometric functions* are given by [4g,5h]

$$\text{isosin}\,\hat{\alpha} = n_2 \times \sin\hat{\alpha}\,, \qquad \text{isocos}\,\hat{\alpha} = n_1 \times \cos\hat{\alpha}\,, \tag{2.52a}$$

$$\text{isosin}^2\hat{\alpha} + \text{isocos}^2\hat{\alpha} = n_1^{-2}\times\text{isocos}^2\hat{\alpha} + n_2^{-2}\times\text{isosin}^2\hat{\alpha} =$$

$$= \cos^2\hat{\alpha} + \sin^2\hat{\alpha} = 1\,, \tag{2.52b}$$

where we have ignored the factorization by the isounit for simplicity. The *isospherical coordinates* can be written [loc. cit.][43]

$$x = r \times n_1 \times \sin(\theta/n_3)\cos(\phi/n_1\times n_2) \tag{2.53a}$$

$$y = r \times n_2 \times \sin(\theta/n_3)\sin(\phi/n_1\times n_2)\,, \tag{2.53b}$$

$$z = r \times n_3 \times \cos(\theta/n_3)\,, \tag{2.53c}$$

We also have the *isopythagorean theorem* for an *isoright isotriangle* with isosides $\hat{A}$ and $\hat{B}$ and isohypothenuse $\hat{D}$ [115]

$$\hat{D}^{\hat{2}} = \hat{D} \hat{\times} \hat{D} = \hat{A}^{\hat{2}} + \hat{B}^{\hat{2}} = \hat{A} \hat{\times} \hat{A} + \hat{B} \hat{\times} \hat{B} \in \hat{R}\,, \tag{2.54}$$

which is trivial on $\hat{E}$ over $\hat{R}$. However, its projection on E over R is not trivial, because it implies the following property among a "triangle" whose sides are curves with only two intersections.

$$\hat{D}^{\hat{2}} = [\, A \times A / n_1{}^2(t, r, \dot{r}, \ldots) + B \times B / n_2{}^2(t, r, \dot{r}, \ldots)\,] \times \hat{1}\,, \tag{2.55}$$

When the isoplane is pseudo-metric with signature (+, –) we can introduce the *isohyperbolic functions* and related property [4g,5h]

$$\text{isocosh}\,\hat{\alpha} = n_1 \times \cosh(\alpha / n_1 \times n_2)\,, \quad \text{isosinh}\,\hat{\alpha} = n_2 \times \sinh(\alpha / n_1 \times n_2)\,, \quad (2.$$

$$\text{isocosh}^{\hat{2}}\,\hat{\alpha} - \text{isosinh}^{\hat{2}}\,\hat{\alpha} = 1, \tag{2.56}$$

As additional elementary isofunctions we have [115]: the *isoexponentiation* (see Sect. 3 for its derivation)

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[43] See monograph [116], Sect. 5.5 with a more general definition of isospherical coordinates with a "hidden" degree of freedom expressed via a free parameter which is absent in the conventional spherical coordinates.

$$\hat{e}^{x} = e_{\xi}^{\hat{x}} = \hat{1} \times \{ e^{\hat{T} \times x} \} = \hat{1} \times \{ e^{\hat{T} \times x} \}, \tag{2.57}$$

where e^x is the ordinary exponentiation.; the *isologarithm* of an isonumber a on isobasis $\hat{e} = e \times \hat{1}$,

$$\hat{\log}_{\hat{e}} \hat{a} = \hat{1} \log_e a . \tag{2.58}$$

with axiom–preserving properties

$$\hat{e}^{\, \mathrm{Isoln}_{\hat{e}} a} = a, \qquad \mathrm{Isoln}\, \hat{e} = \hat{1}, \quad \mathrm{Isoln}\, \hat{1} = 0, \tag{2.59a}$$

$$\mathrm{Isoln}\, (a \hat{\times} b) = \mathrm{Isoln}\, a + \mathrm{Isoln}\, b, \quad \mathrm{Isoln}\, a / b = \mathrm{Isoln}\, a - \mathrm{Isoln}\, b, \tag{2. 59b}$$

$$\mathrm{Isoln}\, a^{-1} = - \mathrm{Isoln}\, a, \qquad \hat{b} \hat{\times} \mathrm{Isoln}\, a = \mathrm{Isoln}\, a^{\, b}, \qquad \text{etc.}; \tag{2.59c}$$

the *isotrace* and *isodeterminant* of an *isomatrix* $\hat{A} = A \times \hat{1}$, where A is an ordinary matrix

$$\mathrm{Isotr}\, \hat{A} = (\mathrm{Tr}\, A) \times \hat{1} \quad \mathrm{Isodet}\, \hat{A} = [\mathrm{Det}\, (A \times T)] \times \hat{1}, \tag{2.60}$$

with axiom–preserving properties

$$\mathrm{Isotr}\, (\hat{A} \hat{\times} \hat{B}) = (\mathrm{Isotr}\, \hat{A}) \hat{\times} (\mathrm{Isotr}\, \hat{B}), \ \mathrm{Isotr}\, (\hat{B} \hat{\times} \hat{A} * \hat{\times} \hat{B}^{\, -\hat{1}}) = \mathrm{Isotr}\, \hat{A}, \tag{2.61}$$

$$\mathrm{Isodet}\, (\hat{A} \hat{\times} \hat{B}) = (\mathrm{Isodet}\, \hat{A}) \hat{\times} (\mathrm{Isodet}\, \hat{B}), \quad \mathrm{Isodet}\, (\hat{A}^{-\hat{1}}) = (\mathrm{Isodet}\, \hat{A})^{-\hat{1}},$$
(6.3.20e)

$$\mathrm{Isodet}\, (\hat{e}^{\hat{A}}) = \hat{e}^{\, \mathrm{Isotr}\, \hat{A}}, \tag{2.62}$$

and others isofunctions the reader can easily construct when needed. For special isofunctions and isotransform, we have to refer the readers to monograph [I15] for brevity.

The *isodual isofunctional isoanalysis* is the image of the isoanalysis under isoduality and inclusion isodual conventional and special isofunctions and isotransforms the reader can easily construct via the isodual map (2.4) applied systematically to the entire theory, including isonumbers, isospaces, etc.

The reader should however be aware that the elaboration of the isotheories with conventional functional analysis (e.g., the use of conventional trigonometry, logarithms, exponentiations, etc.) leads to a number of inconsistencies (such as the

violation of isolinearity) which often remain undetected by the noninitiated reader.

II.2.7: Isotopies and isodualities of differential calculus

Santilli's third important contribution to mathematics has been the identification of an isotopic generalization of the conventional differential calculus first presented at the 1994 *International Workshop on Differential Geometry and Lie Algebras*, held in Thessaloniki, Greece, and then published in ref. [100] (although the new differential calculus is implicitly contained in the first edition of monographs [116] of 1994).

The lifting of the differential calculus has then permitted the achievement of axiomatically consistent the isotopies of virtually all mathematics used in quantitative sciences, including: Newton's equations, analytic and quantum mechanics, differential geometries, etc.

The isodifferential calculus has therefore fundamental relevance for the studies herein considered. In fact, studies on isotopies prior to its appearance in memoir [100] are not invariant and, as such, they are not acceptable on axiomatic as well as physical grounds.

Let $E(x,\delta,R)$ be the ordinary N-dimensional Euclidean space with local coordinates $x = \{x^k\}$, $k = 1, 2, ..., N$, and metric $\delta = \mathrm{diag.}\ (1, 1, 1)$ over the reals $R(n,+,\times)$. Let $\hat{E}(\hat{x},\hat{\delta},\hat{R})$ be its isotopic image with local coordinates $\hat{x} = \{\hat{x}^k\}$ and isometric $\hat{\delta} = \hat{T}\delta$ over the isoreals $\hat{R}(\hat{n},+,\hat{\times})$. Let the isounit be given by the N×N matrix of Class III, $\hat{1} = (\hat{1}_i{}^j) = (\hat{1}_i{}^j) = \hat{T}^{-1} = (T_i{}^j)^{-1} = (T_i{}^j)^{-1}$ whose elements have a smooth but otherwise arbitrary functional dependence on the local coordinates, their derivatives with respect to an independent variable and any needed additional quantity, $\hat{1} = \hat{1}(\hat{x},...)$. The following properties then hold from Definition 2.2:

$$\hat{x}^k = x^k \times \hat{1}\,,\ \hat{x}_k = \hat{\delta}_{ki}\,\hat{x}^i = \hat{T}_k{}^i\,\delta_{ij}\,\hat{x}^j = \hat{T}_k{}^i\,\delta_{ij}\,x^j \times \hat{1} = \hat{T}_k{}^i\,x_i\,,\quad x_i = \delta_{ij}\,x^j,$$

$$\hat{x}^i \hat{\times} \hat{\Delta}_{ij} \hat{\times} \hat{x}^j = (x^i \times \hat{T}_i{}^j \times \delta_{jm} \times x^m) \times \hat{1} = \hat{x}_i \hat{\times} \hat{\Delta}^{ij} \hat{\times} \hat{x}_j = \hat{x}^k \hat{\times} \hat{x}_k = \hat{x}_k \hat{\times} \hat{x}^k\,,\ \hat{\delta}^{ij} = [(\hat{\delta}_{mn})^{-1}]^{ij},$$

$$x^i \times \delta_{ij} \times x^j = x_i \times \delta^{ij} \times x_j = x^i \times x_i = x_i \times x^i\,,\quad \delta^{ij} = [\,(\delta_{mn})^{-1}\,]^{ij}\,. \tag{2.63}$$

Definition 2.5 [100]: *The "first-order isodifferentials" of Class III of the contravariant and covariant coordinates* $\hat{x}^k$ *and* $\hat{x}_k$, on an *isoeuclidean space* $\hat{E}$ *equipped with Kadeisvili's isocontinuity are given by*

$$\hat{d}\,\hat{x}^k = \hat{1}_k{}^i(t, x, ...)\,d\hat{x}^i\,,\quad \hat{d}\,\hat{x}_k = \hat{T}_k{}^i(t, x, ...)\ d\hat{x}_i\,, \tag{2.64}$$

Let $\hat{f}(\hat{x}) = f(\hat{x})\times\hat{1}$ *be a sufficiently smooth isofunction on a closed domain* $\hat{D}(\hat{x}^k)$ *of contravariant coordinates* $\hat{x}^k$ *on* $\hat{E}$. *Then the "isoderivative" at a point* $\hat{a}^k \in \hat{D}(\hat{x}^k)$ *is given by*

$$\hat{f}'(\hat{a}^k) = \frac{\partial \hat{f}(\hat{x})}{\partial \hat{x}^k}\Big|_{\hat{x}^k=\hat{a}^k} = \hat{T}_k{}^i \frac{\partial f(x)}{\partial x^i}\Big|_{\hat{x}^k=\hat{a}} = \mathrm{Lim}_{\hat{d}\hat{x}^k\to\hat{0}^k} \frac{\hat{f}(\hat{a}^k+\hat{d}\hat{x}^k) - \hat{f}(\hat{a}^k)}{\hat{d}x^k} \tag{2.65}$$

where $\hat{\partial}\hat{f}(\hat{x})\hat{/}\hat{\partial}\hat{x}^k = (\partial\hat{f}(\hat{x})/\partial\hat{x}^k)\times\hat{1}$ *is computed on* $\hat{E}$ *and* $\hat{T}_k{}^i\partial f(x)/\partial x^i$ *is the projection in* E. *The "isoderivative" of a smooth isofunction* $\hat{f}(\hat{x})$ *of the covariant variable* $\hat{x}_k$ *at the point* $\hat{a}_k \in \hat{D}(\hat{x}_k)$ *is given by*

$$\hat{f}'(\hat{a}_k) = \frac{\partial \hat{f}(\hat{x})}{\partial \hat{x}_k}\Big|_{\hat{x}_k=\hat{a}_k} = \hat{1}_k{}^i \frac{\partial f(x)}{\partial x_i}\Big|_{\hat{x}_k=\hat{a}_k} = \mathrm{Lim}_{\hat{d}\hat{x}_k\to\hat{0}_k} \frac{\hat{f}(\hat{a}_k+\hat{d}x_k) - \hat{f}(\hat{a}_k)}{\hat{d}\hat{x}_k} \tag{2.66}$$

The above definition and the axiom-preserving character of the isotopies then permit the lifting of the various aspects of the conventional differential calculus. We here mention for brevity the following isotopies: the *isodifferentials of an isofunction* of contravariant (covariant) coordinates $\hat{x}^k$ ($\hat{x}_k$) on $\hat{E}(\hat{x},\hat{\delta},\hat{R})$ are defined via the isoderivatives according to the respective rules

$$\hat{d}\,\hat{f}(\hat{x})|_{\text{contrav.}} = \frac{\hat{\partial}\hat{f}}{\hat{\partial}\hat{x}^k}\hat{\times}\,\hat{d}\hat{x}^k = \hat{T}_k{}^i\times\frac{\partial f}{\partial x^i}\,\hat{1}^k{}_j\,d\hat{x}^j = [df(\hat{x})]\times\hat{1}\ ,$$

$$\hat{d}\,f(x)|_{\text{covar.}} = \frac{\hat{\partial}\hat{f}}{\hat{\partial}\hat{x}_k}\,\hat{d}\hat{x}_k = \hat{1}^k{}_i\times\frac{\partial f}{\partial x_i}\times\hat{T}_k{}^j\times d\hat{x}_j = [\,df(x)\,]\times\hat{1}\ ; \tag{2.67}$$

an iteration of the notion of isoderivative leads to the *second-order isoderivatives*

$$\frac{\partial^2\hat{f}(\hat{x})}{\partial\hat{x}^{k\,2}} = \hat{T}_k{}^i\,\hat{T}_k{}^j\,\frac{\partial^2 f(x)}{\partial x^i\,\partial x^j}\ ,\qquad \frac{\partial^2\hat{f}(\hat{x})}{\partial\hat{x}_k{}^2} = \hat{1}^k{}_i\,\hat{1}^k{}_j\,\frac{\partial^2 f(x)}{\partial x_i\,\partial x_j}\quad \text{(no sums on k} \tag{2.68}$$

and similarly for isoderivatives of higher order; the *isolaplacian* on $\hat{E}(\hat{x},\hat{\delta},\hat{R})$ is given by

$$\hat{\Delta} = \hat{\partial}_k\,\hat{\times}\,\hat{\partial}^k = \hat{\partial}^i\times\hat{\delta}_{ij}\times\hat{\partial}^j = \partial^i\times\delta_{ij}\times\partial^j = \hat{1}^i{}_k\times\partial^k\times\delta_{ij}\times\partial^j\,, \tag{2.69A}$$

$$\hat{\partial}_k = \partial\hat{/}\hat{\partial}\hat{x}^k,\ \partial_k = \partial/\partial x^k,\ \text{etc.}\,, \tag{269B}$$

and and results to be different than the corresponding expression on a Riemannian space $\Re(x,g,R)$ with metric $g(x) = \hat{\delta}$, $\Delta = \hat{\delta}^{-1/2} \times \partial_i \times \hat{\delta}^{1/2} \times \hat{\delta}^{ij} \times \partial_j$.

A few examples are in order. First note the following properties derived from definitions (2.31) and (2.32),

$$\hat{\partial}\hat{x}^i / \hat{\partial}\hat{x}^j = \hat{\delta}^i{}_j = \delta^i{}_j \times \hat{1}\,, \quad \hat{\partial}\hat{x}_i / \hat{\partial}\hat{x}_j = \hat{\delta}_i{}^j\,, \quad \hat{\partial}\hat{x}_i / \hat{\partial}\hat{x}^j = \hat{T}_i{}^j, \quad \hat{\partial}\hat{x}^i / \hat{\partial}\hat{x}_j = \hat{1}^i{}_j\,. \tag{2.70}$$

Next, we have the simple isoderivatives

$$\frac{\hat{\partial}\,(\hat{x}_k \hat{x}^k)}{\hat{\partial}\hat{x}^r} = \frac{\partial\,(\hat{x}^i \hat{\delta}_{ij} \hat{x}^j)}{\partial x^r} = \hat{T}_r{}^i \times \frac{\partial\,(x^i \delta_{ij} x^j)}{\partial x^i} \times \hat{1} = 2 \times \hat{T}_r{}^i \times x^i \times \hat{1} = 2 \times \hat{x}_r$$

$$\frac{\hat{\partial} \ln \hat{\psi}(\hat{x})}{\hat{\partial}\hat{x}^k} = \hat{T}_k{}^i \times \frac{\partial \ln \psi(x)}{\partial x^i} \times \hat{1} = \frac{1}{\hat{\psi}(\hat{x})} \frac{\partial\,\hat{\psi}(\hat{x})}{\partial\,\hat{x}^k} \times \hat{1}\,, \tag{2.71}$$

and similarly for other cases.

For completeness we mention the (indefinite) *isointegration* which, when defined as the inverse of the isodifferential, is given by

$$\hat{\int} \hat{d}\hat{x} = \int \hat{T}\hat{1}\, d\hat{x} = \int d\hat{x} = \hat{x}\,, \tag{2.72}$$

namely, $\hat{\int} = \int \hat{T}$. Definite isointegrals are formulated accordingly.

The above basic notions are sufficient for our needs at this time. The class of isodifferentiable isofunctions of order m will be indicated $\hat{C}^m$.

The *isodual isodifferential calculus* is the isodual image of the preceding one, in which all quantities and all operations are subject ted to the isodual map.

An important property is that Santilli's isodifferentials and isoderivatives verify the condition of preserving the basic isounit $\hat{1}$. Mathematically, this condition is *necessary* to prevent that a set of isofunctions $\hat{f}(\hat{x})$, $\hat{g}(\hat{x})$, ..., on $\hat{E}(\hat{x},\hat{\delta},\hat{R})$ over the isofield $\hat{R}(\hat{n},+,\hat{\times})$ with isounit $\hat{1}$ are mapped via isoderivatives into a set of isofunctions $\hat{f}'(\hat{x})$, $\hat{g}'(\hat{x})$, ..., defined over a *different* field because of the alteration of the isounit. Physically, the condition is also *necessary* because the unit is a pre–requisite for measurements. The lack of conservation of the unit therefore implies the lack of consistent physical applications.

As an example, the following alternative definition of the isodifferential

$$\hat{d}\hat{x}^k = d\,(\hat{1}^k{}_i\, x^i) = [\,(\partial_i \hat{1}^k{}_r)\, x^r + \hat{1}^k{}_i\,]\; dx^i = \hat{W}^k{}_i\, dx^i\,, \tag{2.73}$$

would imply the alteration of the isounit, $\hat{1} \rightarrow \hat{W} \neq \hat{1}$, thus being mathematically and

physically unacceptable.

Nevertheless, when using isoderivatives on independent isomanifolds, say, isoderivatives on coordinates and time, the above rule does not apply and we have

$$\hat{\partial}_t \hat{\partial}_k \hat{f}(\hat{t}, \hat{x}) = \hat{\partial}_t [\hat{\partial}_k \hat{f}(\hat{t}, \hat{x})] = \hat{\partial}_t [T_k^{\ i}(t, x, \ldots) \partial_i f(t, x)] . \tag{2.74}$$

Additional properties of the isodifferential calculus will be identified during the course of our analysis.

Note that the ordinary differential calculus is local-differential on E. The isodifferential calculus is instead local-differential on $\hat{E}$ but, when projected on E, it becomes *integro-differential* because it incorporates integral terms in the isounit.

II.2.8. Isosymplectic geometry

Another important mathematical contribution made by Santilli has been the identification of a novel formulation of the symplectic geometry, originally submitted in memoir [66] of 1988 under the name of *isosymplectic geometry* and subsequently developed in various works (see the review in [115]) which possesses applications much broader than those of the conventional formulation.

The original construction was based on the *isotopic degrees of freedom of the product*, as outlined in Sect. 1.9. In this section we shall outline the isosymplectic geometry as formulated in Ref. [100] via *the isotopic degrees of freedom of the exterior calculus.* A comparison of the two formulations is instructive to see the advances.

The symplectic geometry (see ref.s [109] for a review and comprehensive literature) is the geometry underlying Lie's theory. Santilli studied the isotopies herein reviewed because no genuine broadening of Lie's theory is possible without a corresponding compatible generalization of the symplectic geometry.

As the reader can see, the conventional and isosymplectic geometries coincide at the abstract, coordinate-free level to such an extent, to require no change in the symbols, and only their broader realization as compared to the conventional one. We therefore have in essence *two* different *realizations* of the same abstract geometric axioms, the isotopic realization being broader and admitting of the conventional realization as a particular case.

Despite this abstract unity, it should be noted that the conventional symplectic geometry in its canonical realization permits a *direct representation* (i.e., a representation in the coordinates of the observer) only of (well behaved) local-differential and Hamiltonian systems. By comparison, the isosymplectic geometry in canonical realization is *directly universal* for all well behaved,

nonlocal-integral and nonhamiltonian systems.

The latter direct universality is important in view of the *physical problematic aspects* caused by the practical use of Darboux's transformation of systems which are nonhamiltonian in the inertial coordinate $b = (x, p)$ of the observer to other coordinates $b' = b'(b) = (x'(x, p), p'(x, p))$ in which the systems become Hamiltonian.

As stressed by Santilli in various publications [110], [115], [100], Darboux's map $b \rightarrow b'(b)$ is necessarily nonlinear and noncanonical. As such, the new coordinates b' are not realizable in actual experiments and, if used for mathematical purposes, they imply the violation of Galilei's and Einstein's special relativity because the transformed systems b' are highly *noninertial*.

The primary meaning of the isosymplectic geometry remains that of being the geometry underlying the Lie-Santilli isotheory. However, in so doing, there is the emergence of an *alternative to Darboux's theorem* in the sense that the isogeometry in isocanonical formulation results to be directly universal, thus capable of representing all systems of the class admitted directly in the inertial frame of the observer without any need of Darboux's map.

In this section we shall the main elements of the isosymplectic geometry in local realization by closely following ref. [100]. Unless otherwise stated, all quantities are assumed to satisfy the needed continuity conditions, e.g., of being of class $\hat{C}^{\infty}$ and all neighborhoods of a point are assumed to be star-shaped or have an equivalent topology. Topological aspects are deferred for simplicity to the next section.

Let $\hat{M}(\hat{E}) = \hat{M}(\hat{E}(\hat{\delta},\hat{R})$ be an n-dimensional Tsagas-Sourlas isomanifold [44,45] on the isoeuclidean space $\hat{E}(\hat{x},\hat{\delta},\hat{R})$ over the isoreals $\hat{R} = \hat{R}(\hat{n},+,\hat{\times})$ with $n\times n$-dimensional isounit $\hat{1} = (\hat{1}^{i}{}_{j})$, $i, j = 1, 2, \ldots, n$, of Kadeisvili Class I and local chart $\hat{x} = \{\hat{x}^k\}$. A *tangent isovector* $\hat{X}(\hat{m})$ at a point $\hat{m} \in \hat{M}(\hat{E})$ is an isofunction defined in the neighborhood $\hat{N}(\hat{m})$ of $\hat{m}$ with values in $\hat{R}$ satisfying the *isolinearity conditions*

$$\hat{X}_m(\hat{\alpha} \hat{\times} \hat{f} + \hat{\beta} \hat{\times} g) = \hat{\alpha} \hat{\times} \hat{X}_{\hat{m}}(\hat{f}) + \hat{\beta} \hat{\times} \hat{X}_{\hat{m}}(\hat{g}),$$

$$\hat{X}_{\hat{m}}(\hat{f} \hat{\times} \hat{g}) = \hat{f}(\hat{m}) \hat{\times} \hat{X}_{\hat{m}}(\hat{g}) + \hat{g}(\hat{m}) \hat{\times} \hat{X}_{m}(\hat{f}), \tag{2.75}$$

for all $\hat{f}, \hat{g} \in \hat{M}(\hat{E})$ and $\hat{\alpha}, \hat{\beta} \in \hat{R}$, where $\hat{\times}$ is the isomultiplication in $\hat{R}$ and the use of the symbol ^ means that the quantities are defined on isospaces.

The collection of all tangent isovectors at $\hat{m}$ is called the *tangent isospace* and denoted $T\hat{M}(\hat{E})$. The *tangent isobundle* is the 2n-dimensional union of all possible tangent isospaces when equipped with an isotopic structure (see below).

The *cotangent isobundle* $T^*\hat{M}(\hat{E})$ is the 2n-dimensional dual of the tangent isobundle with local coordinates $\hat{b} = \{\hat{b}^{\mu}\} = \{\hat{x}^k, \hat{p}_k\}$, $\mu = 1, 2, \ldots, 2n$,. Since $\hat{p}$ is

independent of $\hat{x}$, the isounits of the respective differentials are generally different, i.e., we can have $\hat{\partial}x = \hat{1}dx$ and $\hat{\partial}p = \hat{W}dp$, $\hat{1} \neq \hat{W}$, in which case the total isounit of $T^*\hat{M}(\hat{E})$ is the 2n-dimensional Cartesian product $\hat{1}_2 = \hat{1}\times\hat{W}$.

Since the isomomentum is covariant with isodifferential $\hat{\partial}\hat{p}_k = \hat{T}\times d\hat{p}_k$, Santilli [100] assumes the following particular form of the *isounit of the cotangent isobundle*

$$\hat{1}_2 = (\hat{1}_2{}^{\mu}{}_{\nu}) = \begin{pmatrix} \hat{1}_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & \hat{T}_{n\times n} \end{pmatrix} = \hat{T}_2{}^{-1} = (\hat{T}_{2\mu}{}^{\nu})^{-1} \qquad \hat{1} = \hat{T}^{-1}, \qquad (2.76)$$

where $\hat{1}$ is the isounit of the coordinates $\hat{\partial}x = \hat{1}dx$, and $\hat{T}$ is the isounit of the momenta, $\hat{\partial}p = \hat{T}p = \hat{1}^{-1}dp$. In different terms, we select the particular case in which $\hat{W} = \hat{1}^{-1}$.

An *isobasis* of $T^*\hat{M}(\hat{E})$ is, up to equivalence, the (ordered) set of isoderivatives $\hat{\partial} = \{\partial/\partial\hat{b}^{\mu}\} = \{\hat{T}_{2\mu}{}^{\nu}\partial/\partial b^{\nu}\}$. A generic elements $\hat{X} \in T^*\hat{M}(\hat{E})$, called *vector isofield*, can then be written $\hat{X} = \hat{X}^{\mu}(\hat{m})\,\partial/\partial\hat{b}^{\mu} = \hat{X}^{\mu}\,\hat{T}_{2\mu}{}^{\nu}\partial/\partial b^{\mu}$.

The *fundamental one-isoform* on $T^*\hat{M}(\hat{E})$ is given in the local chart $\hat{b}$ by

$$\hat{\theta} = \hat{R}^{\circ}_{\mu}(\hat{b})\,\hat{\partial}\hat{b}^{\mu} = \hat{R}^{\circ}_{\mu}(b)\,\hat{1}_2{}^{\mu}{}_{\nu}\,d\hat{b}^{\nu} = \hat{p}_k\,\hat{\partial}\hat{x}^k = \hat{p}_k\,\hat{1}^k{}_i\,d\hat{x}^i, \qquad \hat{R}^{\circ} = \{\hat{p}, \hat{0}\}. \quad (2.77)$$

The above expression, which can be written $\hat{\theta} = p\hat{\partial}x = p_i\hat{1}^i{}_j dx^j$ to emphasize the differential origin of the isotopies, should be compared with the originally proposed one-isoform $\hat{\theta} = p\hat{\times}dx = p_k\hat{T}^k{}_i dx^i$ [13] obtained via the isotopic degrees of freedom of the product. The preference of the isodifferential calculus over the isomultiplication is then evident for a geometric unity of the conventional and isotopic formulations.

The space $T^*\hat{M}(\hat{E})$, when equipped with the above one-form, is an *isobundle* denoted $T_1{}^*\hat{M}(\hat{E})$. The *isoexact, nowhere degenerate, isocanonical isosymplectic two-isoform* is given by

$$\hat{\omega} = \hat{\partial}\,\hat{\theta} = \tfrac{1}{2}\,\hat{\partial}\,(\hat{R}^{\circ}{}_{\mu}\,\hat{\partial}\hat{b}^{\mu}) = \tfrac{1}{2}\,\omega_{\mu\nu}\,\hat{\partial}\hat{b}^{\mu}\wedge\hat{\partial}\hat{b}^{\nu} =$$

$$= \hat{\partial}\hat{x}^k \wedge \hat{\partial}\hat{p}_k = \hat{1}^k{}_i\,d\hat{x}^i \wedge \hat{T}_k{}^j\,d\hat{p}_j \equiv d\hat{x}^k \wedge d\hat{p}_k. \qquad (2.78)$$

The isomanifold $T^*\hat{M}(\hat{E})$, when equipped with the above two-isoform, is called *isosymplectic isomanifold* in isocanonical realization and denoted $T_2{}^*\hat{M}(\hat{E})$. The *isosymplectic geometry* is the geometry of the isosymplectic isomanifolds.

The last identity in (3.4) show that *the two-isoform $\hat{\omega}$ formally coincides*

with the conventional symplectic canonical two-form ω, and this illustrates the selection of isounit (3.2). The abstract identity of the symplectic and isosymplectic geometries is then evident. However, one should remember that: the underlying metric is isotopic; $\hat{p}_k = \hat{T}_k{}^i p_i$, where p_i is the variable of the conventional canonical realization of the symplectic geometry; and identity $\hat{\omega} = \omega$ no longer holds for the more general isounits $\hat{1}_2 = \hat{1} \times \hat{W}, \hat{1} \neq \hat{W}^{-1}$.

Note that the *isosymplectic geometry has the Tsagas-Sourlas Integro-differential topology* and, as such, it can characterize interior systems when all nonlocal-integral terms are embedded in the isounit.

A *vector isofield* $\hat{X}(\hat{m})$ defined on the neighborhood $\hat{N}(\hat{m})$ of a point $\hat{m} \in \hat{T}_2{}^*\hat{M}(\hat{E})$ with local coordinates $\hat{b}$ is here called a (local) *Hamilton-Santilli isofield* when there exists an isofunction $\hat{H}$ on $\hat{N}(\hat{m})$ over $\hat{R}$ such that

$$\hat{X} \lrcorner \hat{\omega} = \hat{d}\,\hat{H}, \quad \text{i.e.,}$$

$$\omega_{\mu\nu}\, \hat{X}^{\nu}(\hat{m})\, \hat{d}\hat{b}^{\mu} = \hat{d}\hat{H}(\hat{m}) = (\,\hat{\partial}\hat{H} \,/\, \hat{\partial}\hat{b}^{\mu})\, \hat{d}\hat{b}^{\mu}, \tag{2.79}$$

We are now equipped to present the main result of paper [100], Santilli's alternative to Darboux's Theorem for the representation of nonlinear, nonlocal-integral and nonhamiltonian interior systems within the fixed coordinates of their experimental observation, which can be formulated as follows.

Theorem 2.1 (Direct Universality of the Isosymplectic Geometry for Interior Systems [100]): *Under sufficient continuity and regularity conditions, all possible vector fields which are not (locally) Hamilton in the given coordinates are always Hamilton-Santilli in the same coordinates, that is, there exists a neighborhood* $\hat{N}(\hat{m})$ *of a point* $\hat{m}$ *of their variable* $\hat{b} = (\hat{x}, \hat{p})$ *under which Eq.s (3.5) hold.*

Proof. Let $\hat{X}^{\mu}(\hat{b})$ be a vector field which is nonhamiltonian in the chart $\hat{b}$, and consider the decomposition

$$\hat{X}(b) = \hat{\Gamma}^{\mu}{}_{\alpha}(b)\, \hat{X}_o{}^{\alpha}(b)\,, \tag{2.80}$$

where the 2n×2n matrix $(\hat{\Gamma}^{\mu}{}_{\alpha})$ is nowhere degenerate and $\hat{X}_o{}^{\alpha}$ is the maximal, local-differential and Hamiltonian sub-vector field, i.e., there exists a function H(b) and a neighborhood N(m) of a point m of b = (x, p) such that

$$\omega_{\alpha\beta}\, \hat{X}_o{}^{\beta}(m)\, db^{\alpha} = dH(m) = (\,\partial H \,/\, \partial\, b^{\alpha}\,)\, \hat{d}\hat{b}^{\alpha}, \tag{2.81}$$

and all nonlocal-integral and nonhamiltonian terms are embedded in $\hat{\Gamma}$. Then, there

always exists an isotopy such that

$$\omega_{\mu\nu}\, \hat{X}^{\nu}(\hat{m})\, d\hat{b}^{\mu} = \omega_{\mu\alpha}\, \hat{T}^{\alpha}{}_{\beta}(\hat{m})\, \hat{X}_{0}{}^{\beta}(\hat{m})\, d\hat{b}^{\mu} =$$

$$= d\hat{H}(\hat{m}) = (\partial\hat{H} / \partial\hat{b}^{\mu})\, d\hat{b}^{\mu} = \hat{T}_{\mu}{}^{\beta}\, \partial H / \partial b^{\beta})\, db^{\mu}. \qquad (2.82)$$

In fact, the script $\hat{X}^{\mu}$ is only a unified formulation in 2n dimension of two separate terms each in n-dimension. Therefore, the quantity $\hat{T}$ has the structure

$$\hat{T} = \begin{pmatrix} \hat{A}_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & \hat{B}_{n\times n} \end{pmatrix}. \qquad (2.83)$$

The identification

$$\hat{1} = \begin{pmatrix} \hat{B}^{-1}{}_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & \hat{A}^{-1}{}_{n\times n} \end{pmatrix}, \qquad (2.84)$$

then implies

$$\hat{1}^{\mu}{}_{\alpha}\, \omega_{\mu\nu}\, \hat{T}^{\nu}{}_{\rho} \equiv \omega_{\alpha\rho}, \qquad (2.85)$$

and identities (3.8) always exist. **q.e.d.**

Corollary 2.1.A: *For all Newtonian systems we have* $\hat{A} = \hat{B}^{-1}$, *i.e. , the 2n-dimensional isounit of the cotangent isobundle has structure* (2.90).

Proof. All Newtonian systems in the 2n-dimensional, first-order, vector field form can be written in disjoint n-component

$$\begin{pmatrix} dx/dt \\ dp/dt \end{pmatrix} = \begin{pmatrix} p/m \\ F^{SA} + F^{NSA} \end{pmatrix} = \hat{X}(b) = (\hat{X}^{\mu}(b)) \qquad (2.86)$$

where SA (NSA) stands for variational selfadjointness (nonselfadjointness), i.e., the integrability conditions for the existence (lack of existence) of a Hamiltonian [109]. Thus $F^{SA} = -\partial H/\partial x$, with $H = p^2/2m + V(x)$, while there is no such Hamiltonian for F^{NSA}.

Then, the isohamiltonian representation explicitly reads

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p/m \\ F^{SA} + F^{NSA} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} p/m \\ F^{SA} \end{pmatrix} =$$

$$= \begin{pmatrix} -B\,F^{SA} \\ A\,p/m \end{pmatrix} = \begin{pmatrix} \partial H/\partial x \\ \partial H/\partial p \end{pmatrix} = \begin{pmatrix} B\,\partial H/\partial x \\ A\,\partial H/\partial p \end{pmatrix}. \tag{2.87}$$

From which we have the general solution

$$\hat{1} = B = 1 + F^{NSA}/F^{SA} = A^{-1} = \hat{T}^{-1}, \tag{2.88}$$

where the last identity follow from the fact that, since $\partial H/\partial p = p/m$, A remains arbitrary and can be therefore assumed to be $A = B^{-1}$. **q.e.d.**

The above results confirm, this time on independent geometric grounds, the corresponding results achieved in Sect. 2.5 on analytic grounds, thus confirming the overall unity of isotopic methods.

Santilli [100] completes his study by showing that the above geometric isotopies do indeed preserve the remaining axiomatic properties of the symplectic geometry. For this it is sufficient to prove the preservation under isotopies of the Poincaré Lemma and of Darboux's Theorem.

To prove the preservation of the Poincaré Lemma one can easily construct isoforms $\hat{\Phi}_p$ of arbitrary order p. The proof of the following property is a simple isotopy of the conventional proof (see, e.g., [20]) via the use of the isodifferential calculus.

Lemma 2.1 (Poincare´– Santilli Lemma [100]): *Under the assumed smoothness and regularity conditions, isoexact p–isoforms are isoclosed, i.e.,*

$$\hat{d}\,\hat{\Phi}_p = \hat{d}\,(\hat{d}\hat{\phi}_{p-1}) \equiv 0. \tag{2.89}$$

The nontriviality of the above result is illustrated by the following

Corollary 2.1.A: *Isoexact p–isoform are not necessarily closed, i.e., their projection in the original tangent bundle does not necessarily verify the Poincaré Lemma.*

By comparison, we should mention that the original formulation of the isopoincaré lemma [32,37], that via the isotopic degrees of freedom of the product, did verify the Poincaré lemma in both the conventional and isotopic bundle.

To prove the preservation of the Darboux's Theorem [9], consider the *general one–isoform* in the local chart $\hat{b}$

$$\hat{\theta}(\hat{b}) = \hat{R}_\mu(\hat{b})\,\hat{\partial}\hat{b}^\mu = \hat{R}_\mu(b)\,\hat{1}_2{}^\mu{}_\nu(t, b, db/dt, ...)\,db^\nu , \tag{2.90}$$

where

$$\hat{R} = \{\hat{P}(\hat{x}, \hat{p}), \hat{Q}(\hat{x}, \hat{p})\} . \tag{2.91}$$

The *general isosymplectic isoexact two-isoform* in the same chart is then given by

$$\hat{\Omega}(\hat{b}) = \tfrac{1}{2}\,\hat{\partial}\,(\hat{R}_\mu(\hat{b})\,\hat{\partial}\hat{b}^\nu) = \tfrac{1}{2}\,\hat{\Omega}_{\mu\nu}(t, \hat{b}, \partial\hat{b}/\partial t, ...)\,\hat{\partial}\hat{b}^\mu \wedge \hat{\partial}\hat{b}^\nu ,$$

$$\hat{\Omega}_{\mu\nu} = \frac{\hat{\partial}\hat{R}_\nu}{\hat{\partial}\hat{b}^\mu} - \frac{\hat{\partial}\hat{R}_\mu}{\hat{\partial}\hat{b}^\nu} = \hat{T}_{2\mu}{}^\alpha\,\frac{\partial\hat{R}_\nu}{\partial\hat{b}^\alpha} - \hat{T}_{2\nu}{}^\alpha\,\frac{\partial\hat{R}_\mu}{\partial\hat{b}^\alpha} . \tag{2.92}$$

One can see that, while at the canonical level the exact two-form ω and its isotopic extension $\hat{\omega}$ formally coincide, *this is no longer the case for exact, but arbitrary two forms* Ω *and* $\hat{\Omega}$ *in the same local chart.*

Note that the isoform $\hat{\Omega}$ is isoexact, $\hat{\Omega} = \hat{\partial}\hat{\theta}$, and therefore isoclosed, $\hat{\partial}\hat{\Omega} = 0$ (Lemma 3.1), in isospace over the isofield $\hat{R}$. However, if the same isoform $\hat{\Omega}$ is projected in ordinary space and called Ω, it is no longer necessarily exact, $\Omega \neq d\theta$ and, therefore, it is not generally closed, $d\Omega \neq 0$.

Recall that the Poincaré Lemma $d\Omega = d(d\theta) = 0$ for the case of Birkhoffian two-form Ω (Sect. 2.4) provides the necessary and sufficient conditions for the tensor $\Omega^{\mu\nu} = [\,(\Omega_{\alpha\beta})^{-1}]^{\mu\nu}$ to be Lie [110]. It is easy to prove that this basic property persists under isotopy, although it characterizes the broader Lie-Santilli isotheory. We therefore have the following

Theorem 2.2 (General Lie-Santilli Brackets [100]): *Let* $\hat{\Omega}(\hat{b}) = \hat{\partial}\hat{\theta} = \hat{\partial}(\hat{R}_\mu\hat{\partial}\hat{b}^\mu) = \hat{\Omega}_{\mu\nu}\hat{\partial}\hat{b}^\mu \wedge \hat{\partial}\hat{b}^\nu$ *be a general exact two-isoform. Then the brackets among sufficiently smooth and regular isofunctions* $\hat{A}(\hat{b})$ *and* $\hat{B}(\hat{b})$ *on* $T_2{}^*M(\hat{E})$

$$[\,\hat{A}, \hat{B}\,]_{\text{isot.}} = \frac{\hat{\partial}\hat{A}}{\hat{\partial}\hat{b}^\mu}\,\hat{\Omega}^{\mu\nu}\,\frac{\hat{\partial}\hat{B}}{\hat{\partial}\hat{b}^\nu} ,$$

$$\hat{\Omega}^{\mu\nu} = \left[\left(\frac{\hat{\partial}\hat{R}_\alpha}{\hat{\partial}\hat{b}^\beta} - \frac{\hat{\partial}\hat{R}_\beta}{\hat{\partial}\hat{b}^\alpha}\right)^{-1}\right]^{\mu\nu} , \tag{2.93}$$

satisfy the Lie-Santilli axioms (Sect. II.3.3) *in isospace (but not necessarily the*

same axioms when projected in ordinary spaces).

The above theorem establishes that *the isosymplectic geometry is indeed the geometry underlying the Lie-Santilli isotheory,* as discussed in more details in the accompanying paper by Kadeisvili [16]. In particular, the isocanonical two-isoform characterizes the isocanonical realization of the Lie-Santilli brackets studied in the next section, while brackets (2.93) are the most general possible ones.

Even though we cannot use Darboux's theorem in practical applications for the reasons indicated earlier, it is nevertheless important for completeness to prove that it admits a simple yet significant isotopies.

Theorem 2.3 (Darboux-Santilli Theorem): *A 2n-dimensional cotangent isobundle $T_2{}^*\hat{M}(\hat{E})$ equipped with a nowhere degenerate, exact, $\hat{C}^{\infty}$ two-isoform $\hat{\Omega}$ in the local chart $\hat{b}$ is an isosymplectic manifold if and only if there exist coordinate transformations $\hat{b} \to \hat{b}'(\hat{b})$ under which $\hat{\Omega}$ reduces to the isocanonical two-isoform $\hat{\omega}$, i.e.,*

$$\frac{\partial \hat{b}^{\mu}}{\partial \hat{b}'^{\,\alpha}}\, \hat{\Omega}_{\mu\nu}(\hat{b}(\hat{b}'))\, \frac{\partial \hat{b}^{\nu}}{\partial \hat{b}'^{\,\beta}} = \omega_{\alpha\beta}\,. \tag{2.94}$$

Proof. Suppose that the transformation $\hat{b} \to \hat{b}'(b)$ occurs via the following intermediate transform $\hat{b} \to \hat{b}''(\hat{b}) \to \hat{b}'(b''(b))$. Then there always exists a transform $\hat{b} \to \hat{b}''$ such that

$$(\partial \hat{b}^{\rho} / \partial \hat{b}''^{\sigma})(\hat{b}'') = \hat{1}^{\rho}{}_{\sigma}(\hat{b}(\hat{b}'')), \tag{2.95}$$

under which the general isosymplectic tensor $\hat{\Omega}_{\mu\nu}$ reduces to the Birkhoffian form when recompute in the $\hat{b}$ char

$$\frac{\partial \hat{b}^{\mu}}{\partial \hat{b}''^{\,\alpha}}\, \hat{\Omega}_{\mu\nu}(\hat{b}(\hat{b}''))\, \frac{\partial \hat{b}^{\nu}}{\partial \hat{b}''^{\,\beta}}\Big|_{\hat{b}''} = \left(\frac{\partial \hat{R}_{\nu}}{\partial \hat{b}^{\alpha}} - \frac{\partial \hat{R}_{\mu}}{\partial \hat{b}^{\nu}} \right)\Big|_{\hat{b}''} = \Omega_{\alpha\beta}\,\Big|_{\hat{b}''}\,. \tag{2.96}$$

The existence of a second transform $\hat{b}'' \to \hat{b}'$ reducing $\Omega_{\alpha\beta}$ to $\omega_{\alpha\beta}$ is then known to exist (see, e.g., [30]). This proves the necessity of the isodarboux transform. The sufficiency is proved as in the conventional case [109]. **q.e.d.**

The nonlinear, nonlocal and noncanonical character of the isotopies is evident from the preceding analysis. It is important to point out that linearity,

locality and canonicity are reconstructed in isospace over isofields, as studied later on.

The isotopies of the remaining aspects of the symplectic geometry (Lie derivative, global treatment, symplectic group, etc.) can be constructed along the preceding lines and are omitted for brevity. The isosymplectic geometry is also expected to admit a *genotopic* and *hyperstructural* extensions although they are not studied in here for brevity (see [100,101]).

We should *mention* that the preceding formulation of the isosymplectic geometry is solely restricted for the representation of *matter*. The characterization of antimatter is made via the antiautomorphic isodual map $\hat{1}_2 \to \hat{1}_2{}^d = -\hat{1}_2$. This permitted Santilli the discovery of the *isodual isosymplectic geometry* [100] which is characterized by *isodual coordinates* $\hat{b}^d$, *isodual isodifferentials* $\hat{\partial}^d\hat{b}^d$, *isodual one-isoforms* $\hat{\theta}^d(\hat{b}^d)$, *isodual two-isoforms* $\hat{\omega}^d$, *isodual cotangent isobundle* $T^*\hat{M}^d(\hat{E}^d)$, and similar isodualities.

One should keep in mind that the isodual isosymplectic geometry admits as a particular case *Santilli's isodual symplectic geometry,* which is a novel anti-isomorphic image of the conventional geometry.

The proof of the following property is instructive

Lemma 2.2 [100,101]: *Isosymplectic one-, two- and p-isoforms are isoselfdual, i.e., invariant under isoduality.*

The above lemma establishes that the mathematical structure of the conventional symplectic geometry is applicable for the characterization of both, particles and antiparticles when formulated in spaces and their isoduals, respectively.

Note that the use of the *conventional* symplectic geometry for the characterization of *antiparticles* leads under symplectic quantization to the physical inconsistencies recalled earlier (an operator image which is not the charge conjugate particle, but merely a particle with a change in the sign of the charge). The study of other aspects is left to the interested reader.

In closing we mention the remarkable abstract unity of the conventional symplectic geometry and Santilli's isosymplectic geometry and its isodual which could be all expressed with the *same abstract symbols,* which are then differentiated via different *realizations.*

In short, Santilli has proved that, contrary to a rathe popular belief in mathematical circles, the contemporary formulation of the symplectic geometry is far from being the most general one, because it admits the isosymplectic formulation, as well as the yet broader genosymplectic and hypersymplectic realizations [100,101].

II.2.9: Isotopies and isodualities of Newtonian mechanics.

As it is well known (see, e.g. [13]), Lie's theory admits two fundamental realizations, one in classical and one in quantum mechanics, with interconnecting map given by the naive or symplectic quantization.

The preceding isotopies were introduced by Santilli for the construction of step-by-step isotopic generalizations of classical [62] and quantum [61] mechanics and their interconnecting maps. The new mechanics have been conceived for the most general possible, nonlinear, nonlocal and noncanonical, interior dynamical problems. They reached maturity of formulation only recently in memoir [100] following the advent of the isodifferential calculus.

It is important to review at least the essential elements of the isotopic classical and operator mechanics because they provide corresponding realizations of the Lie-Santilli isotheory used in applications. As a matter of fact, Santilli proposed the isotopies of Lie's theory precisely for quantitative treatments of the above generalized mechanics.

To conduct our outline, we shall keep using Santilli's notation ([100], [115]) of putting a "hat" on all quantities belonging to isotopic formulations, while conventional symbols are used for quantities belonging to conventional formulations (see [72] for details).

As it is well known, conventional classical mechanics is formulated in the configuration space via the seven-dimensional space $E(t,\delta,R)\times E(x,\delta,R)\times E(v,\delta,R)$ where t is time, $x = \{x^k\}$ represents the space coordinates and $v = \{v^k\}$ represents the velocities, the latter being independent from the former.

The Class I isotopies of classical mechanics in configuration space require their formulation in the isospace

$$\hat{S}(\hat{t},\hat{x},\hat{v}) = \hat{E}(\hat{t},\hat{R}_t) \hat{\times} \hat{E}(\hat{x},\hat{\delta},\hat{R}) \times \hat{E}(\hat{v},\hat{\delta},\hat{R}), \tag{2.97}$$

characterized by the total isounit

$$\hat{1}_{tot} = \hat{1}_t \times \hat{1}_x \times \hat{1}_v, \tag{2.98}$$

where: $\hat{1}_t = T_t^{-1}$ is the (one-dimensional) *isounit of time* and $\hat{1}_x = \hat{T}^{-1}$ is the (three-dimensional) *isounit of space* and $\hat{1}_v$ is the three-dimensional *isounit of the velocities* hereon assumed to coincide with $\hat{1}_r$ for simplicity. By assuming that the isotime is contravariant we have $\hat{t} \equiv t\times\hat{1}_t$, while for the space components we have the general rules

$$\hat{x} = \{x^k\} \times \hat{1} \equiv \{x^k\}, \qquad \hat{x}_k = \hat{\delta}_{ki}\, \hat{x}^i = \hat{T}_k{}^i \times \delta_{ij} \times \hat{x}^j = \hat{T}_k{}^i\, x_i \times \hat{1},$$

$$\hat{v} = \{v^k\} \times \hat{1} \equiv \{dx^k/dt\} \times \hat{1}, \qquad \hat{v}_k = \hat{\delta}_{ki}\, \hat{v}^i = \hat{T}_k{}^i\, v_i \times \hat{1}. \tag{2.99}$$

The isodifferential calculus on $\hat{S}(\hat{t},\hat{x},\hat{v})$ is then based on the following *space and time isodifferentials* and *isoderivatives*,

$$\hat{d}\hat{t} = \hat{1}_t \times d\hat{t}, \quad \hat{d}\hat{x}^k = \hat{1}^k{}_i \times d\hat{x}^i, \qquad \hat{d}\hat{x}_k = \hat{T}_k{}^i \times d\hat{x}_i,$$

$$\hat{d}\hat{v}^k = \hat{1}^k{}_i \times dv^i, \qquad \hat{d}\hat{v}_k = \hat{T}_k{}^i \times d\hat{v}_k,$$

$$\hat{\partial}/\hat{\partial}\hat{t} = \hat{T}_t\, d/dt, \quad \hat{\partial}/\hat{\partial}\hat{x}^k = \hat{T}_k{}^i \times \partial/\partial x^i, \quad \hat{\partial}/\hat{\partial}\hat{x}_k = \hat{1}^k{}_i \times \partial/\partial\hat{x}_i,$$

$$\hat{\partial}/\hat{\partial}\hat{v}^k = \hat{T}_k{}^i \times \partial/\partial v^i, \quad \hat{\partial}/\hat{\partial}\hat{v}_k = \hat{1}^k{}_i \times \partial/\partial\hat{v}_i \tag{2.100}$$

with basic properties (where we shall ignore hereon for simplicity of notation the isoquotients and the consequential final multiplication by $\hat{1}$)

$$\hat{\partial}\hat{x}^i / \hat{\partial}\hat{x}^j = \delta^i{}_j, \quad \hat{\partial}\hat{x}_i / \hat{\partial}\hat{x}_j = \delta_i{}^j, \quad \hat{\partial}\hat{x}^i / \hat{\partial}\hat{x}_j = \hat{1}^i{}_j, \quad \hat{\partial}\hat{x}_i / \hat{\partial}\hat{x}^j = \hat{T}_i{}^j,$$

$$\hat{\partial}\, (\hat{v}^i\, \delta_{ij}\, \hat{v}^j) / \hat{\partial}\hat{v}^k = 2\hat{v}_k. \tag{2.101}$$

The fundamental dynamical equations are then given by the *Newton–Santilli isoequations* on $\hat{S}(\hat{t},\hat{r},\hat{v})$, first submitted by Santilli in memoir [100], p. 31, Eq.s (2.5),

$$\hat{m} \hat{\times} \frac{\hat{d}v_k}{\hat{d}\hat{t}} - \frac{\hat{\partial}\, \hat{U}_k(\hat{t},\hat{r})}{\hat{\partial}\hat{r}^i}\, \frac{\hat{d}\hat{r}^i}{\hat{d}\hat{t}} + \frac{\hat{\partial}\hat{U}_o(\hat{t},\hat{r})}{\hat{\partial}\hat{r}^k} \stackrel{A}{=} \tag{2.102}$$

$$= \hat{T}_k{}^i \left\{ m \frac{dv_i}{dt} - \frac{\partial\, U_i(t, r)}{\partial r^s}\, \frac{dx^s}{dt} + \frac{\partial\, U_o(t, r)}{\partial r^i} - F^{NSA}{}_i(t, r, v) \right\} = 0$$

where $\hat{m} = m \times \hat{1}_t$ is the *isomass*, i.e., the mass in isospace, and NSA stands for *variational nonselfadjointness*, i.e., the violation of Helmholtz's integrability conditions for the existence of a potential (see volumes [109] for detailed studies).

The first main function of the above equations is to turn dynamical equations which *do not* admit a Lagrangian or a Hamiltonian representation in the (t, r, v) coordinates of the experimenter into an isotopic form which is indeed representable with the Lagrange' and Hamilton's equations on isospaces over

isofields. This objective is achieved by embedding all nonpotential forces in the *differentials*, i.e., representing the deviations from the geometry of empty space by the isogeometry.

The second main objective of Eq.s (2.102) is to lift the historical Newtonian representation of "massive points" into the representation of *expended, nonspherical and deformable bodies* with a shape represented precisely by the $\hat{1}$-matrix, e.g., with explicit diagonal form for spheroidal ellipsoids

$$\hat{1} = \mathrm{Diag.}\,(n_1^2, n_2^2, n_3^2), \tag{2.103}$$

with non diagonal expression for more complex shapes.

The third main objective of Eq.s (2.102) is to extend the strictly "local-differential" character of the historical equations (as necessary from the underlying Euclidean topology) into a form admitting of "nonlocal-integral interactions", i.e., interactions representable with surface or volume integrals, as typically occurs for resistive forces, and as permitted by the Tsagas-Sourlas-Santilli integro-differential topology, provided that all integral terms are embedded in the isounits.

Recall that the actual size and shape of a body has no impact in its dynamical evolution when moving in vacuum. This is not the case when the same body moves within resistive media, where the size and shape of the body directly affect its trajectory.

The new class of systems represented by Eq.s (2.102) is given by extended, nonspherical and deformable bodies moving within resistive media whose center of mass trajectory is conventional, i.e., local-differential, while admitting integral corrective terms due to the shape.

As a specific example, consider an originally spherical body of mass m which moves along the x-axis within a resistive medium (say, gas or liquid) by acquiring an ellipsoidical shape σ with semiaxes (a^2, b^2, c^2). By ignoring potential forces for simplicity, suppose that the body experiences only a nonlocal-integral resistive force of the type $F_x^{NSA} = -\gamma v_x^2 \int_\sigma d\sigma\, \mathcal{F}(\sigma, \ldots)$, where NSA stands for variational nonselfadjointness [49], $\gamma > 0$ and $\mathcal{F}$ is a suitable kernel. The above systems can be directly represented in isoconfiguration space $\hat{S}(\hat{t}, \hat{x}, \hat{v})$ via the Newton-Santilli equation

$$m \hat{\times} \hat{\partial} \hat{v}_x \,\hat{/}\, \hat{\partial} \hat{t} = 0, \text{ i. e., } m \times d\,(\hat{T}_x^{\,x} v_x)\,/\,dt = \tag{2.104}$$

$$= \hat{T}_x^{\,x} \times [\, m \times dv_x / dt + m \times \hat{1}^x_{\ x}\,(d\,\hat{T}_x^{\,x} / dt) \times v_x \,] = 0,$$

$$\hat{m} = m, \quad \hat{1}_t = 1, \quad \hat{1}_x^{\,x} = \mathrm{diag.}\,(a^{-2}, b^{-2}, c^{-2})\, \exp\{-\gamma\, t\, v_x \int_\sigma d\sigma\, \mathcal{F}(\sigma, \ldots)\}.$$

The interested reader can then construct a virtually endless number of other examples. Note that, by comparison, the conventional Newton's equations can only represent *point-like particles under local-differential interactions*. By recalling that the terms "Newtonian mechanics" are referred to point-particles under local-differential interactions, the emerging new mechanics for extended-deformable particles under integro-differential interactions shall be referred to as the *Newton-Santilli isomechanics*.

Recall that the notion of isoduality of Sect. 2.1 also applies to conventional formulations, including conventional mechanics. This has permitted Santilli to identify new anti-isomorphic images of conventional Newtonian, Lagrangian and Hamiltonian mechanics for the representation of antimatter first submitted in memoir [5g].

The *isoduality of the ordinary Newton's equations* are defined on the *isodual space*

$$S^d(t^d,x^d,v^d) = E^d(t^d,R^d{}_t)\times E^d(x^d,\delta^d,R^d)\times E^d(v^d,\delta^d,R^d)\,, \qquad (2.105)$$

with *isodual unit*

$$1^d{}_{tot} = 1^d{}_t \times 1^d \times 1^d, \qquad 1_t{}^d = -1, \qquad 1^d = \mathrm{diag.}\,(-1,-1,-1), \qquad (2.106)$$

yielding the *Newton-Santilli isodual equation* (Ref. [100], p. 39, Eq.s (2.23),

$$m^d x^d \frac{d^d v_k{}^d}{d^d t^d} - \frac{d^d}{d^d t^d} \frac{\partial^d U^d(t^d, x^d, v^d)}{\partial^d v^{kd}} + \frac{\partial^d U^d(t^d, x^d, v^d)}{\partial^d x^{kd}} = 0\,. \qquad (2.107)$$

are are used in this study for the representation of antimatter in *exterior* conditions in vacuum.

It is an instructive exercise for the interested reader to prove that *Newton's equations change sign under isoduality* (this requires the isoduality not only of all multiplications, but also of all quotients). However, such a *negative* value is referred to a *negative* unit, thus establishing their equivalence to the *positive* value of the conventional equations referred to *positive* units. Note that under the above representation, antimatter possesses *negative masses*, and *moves backward in time*.

It has been shown in Ref.s [105,106] that Eq.s (2.61) represent the totality of the experimental data on the *classical* behavior of antiparticles.

The *isodual isonewton equations* are defined don the *isodual isospace*

$$\hat{S}^d(\hat{t}^d,\hat{x}^d,\hat{v}^d) = \hat{E}^d(\hat{t}^d,\hat{R}^d{}_t)\times\hat{E}^d(\hat{x}^d,\hat{\delta}^d,\hat{R}^d)\times\hat{E}^d(\hat{v}^d,\hat{\delta}^d,\hat{R}^d)\,, \qquad (2.108)$$

with *isodual isounit*

$$\hat{1}^d{}_{tot} = \hat{1}^d{}_t \times \hat{1}^d \times \hat{1}^d, \qquad \hat{1}_t{}^d = -\hat{1}_t, \qquad \hat{1}^d = -\hat{1}, \tag{2.109}$$

can be written (Ref. [5g], p. 39, Eq.s (2.24),

$$\hat{m}^d \hat{\times}^d \frac{\hat{\partial}^d \hat{v}_k{}^d}{\hat{\partial}^d \hat{t}^d} - \frac{\hat{\partial}^d}{\hat{\partial}^d \hat{t}^d} \frac{\hat{\partial}^d \hat{U}^d(\hat{t}^d, \hat{x}^d, \hat{v}^d)}{\hat{\partial}^d \hat{v}^{kd}} + \frac{\hat{\partial}^d \hat{U}^d(\hat{t}^d, \hat{x}^d, \hat{v}^d)}{\hat{\partial}^d \hat{x}^{kd}} = 0 . \tag{2.110}$$

and are used in this study for the treatment of antimatter in interior conditions.

For all details and examples of the emerging *Newton-Santilli isomechanics*, we refer the interested reader for brevity to memoir [100].

II.2.10. Isotopies and isodualities of Lagrangian mechanics.

In memoir [100] Santilli has shown that all possible isoequations (2.56) admit a *direct analytic representation*, that is, a representation via a first-order action functional in isospace over isofields (universality), directly in the coordinates of the experimenter (direct universality).

First, we note the following:

Theorem 2.4 (Direct Universality of first-order isoactions) [100]: *Under sufficient smoothness and regularity conditions, all possible action functionals of arbitrary (finite) order in the Euclidean space* $S(t,x,v) = E(t,R_t) \times E(x,\delta,R) \times E(v,\delta,R)$ *can always be identically rewritten as first-order action isofunctionals of Class I (or isoaction) on isospace* $\hat{S}(\hat{t},\hat{x},\hat{v}) = \hat{E}(\hat{t}) \times \hat{E}(\hat{x},\hat{\delta},\hat{R}) \times \hat{E}(\hat{v},\hat{\delta}\hat{R})$ *in the coordinate of the observer*

$$\hat{A} = \int_{t_1}^{t_2} dt\, \mathcal{L}(t, r, v, a, \ldots) = \hat{\int}_{\hat{t}_1}^{\hat{t}_2} \hat{d}\hat{t}\, \hat{L}(\hat{t}, \hat{r}, \hat{v}),$$

$$\hat{L} = \hat{\tfrac{1}{2}} \hat{\times} \hat{m} \hat{\times} \hat{v}_k \hat{\times} \hat{v}^k - \hat{U}^k(\hat{t}, \hat{r}) \hat{\times} \hat{v}_k - \hat{U}_o(\hat{t}, \hat{r}) \tag{2.111}$$

In fact, the above identity is overdetermined because, for each given $\mathcal{L}$, there exist infinitely many choices of $\hat{m}$, $\hat{T}_t$, $\hat{T}_i^j$, $\hat{U}_k$ and $\hat{U}_o$. We shall assume that integral terms are admitted in the integrand provided that they are all embedded in the isometric to be compatible with the Tsagas-Sourlas isotopology.

The *isovariational calculus* is a simple extension of the isodifferential

calculus (see memoir [5100 for details). The isotopies of the historical works by Euler's [17b] and Lagrange [17c] then lead to the following:

Theorem 2.5 (Euler-Santilli Necessary Condition) [loc. cit.]: *A necessary condition for a Class I isodifferentiable isopath $\hat{P}_0$ in isospace $\hat{S}(\hat{t},\hat{x},\hat{v})$ to be an extremal of the action isofunctional $\hat{A}$ of the same class is that all the following equations* (Ref. [100], p. 44, Eq.s (2.41)

$$\hat{L}_k(\hat{P}_0) = \left\{ \frac{\hat{\partial}}{\hat{\partial}\hat{t}} \frac{\hat{\partial}\,\hat{L}(\hat{t},\hat{r},\hat{v})}{\hat{\partial}\,\hat{v}^k} - \frac{\hat{\partial}\,\hat{L}(\hat{t},\hat{r},\hat{v})}{\hat{\partial}\,\hat{r}^k} \right\}(\hat{P}_0) = 0, \qquad (2.112)$$

are identically verified along $\hat{P}_0$.

Along lines similar to those of the current literature, Eq.s (2.66) shall be called *Euler-Santilli isoequations* of Class I when dealing with variational-optimization problems and *Lagrange-Santilli isoequations* of the same class when used for the description of dynamical systems.

When explicitly written, the isoequations (2.66) characterize the the Newton-Santilli isoequations (2.56) according to the identifications

$$\frac{\hat{\partial}}{\hat{\partial}\hat{t}} \frac{\hat{\partial}\,\hat{L}(\hat{t},\hat{r},\hat{v})}{\hat{\partial}\,\hat{v}^k} - \frac{\hat{\partial}\,\hat{L}(\hat{t},\hat{r},\hat{v})}{\hat{\partial}\,\hat{r}^k} =$$

$$= \hat{m}\,\hat{\times}\,\frac{\hat{\partial} v_k}{\hat{\partial}\hat{t}} - \frac{\hat{\partial}\,\hat{0}_k(\hat{t},\hat{r})}{\hat{\partial}\,\hat{r}^i}\,\frac{\hat{\partial}\,\hat{r}^i}{\hat{\partial}\hat{t}} + \frac{\hat{\partial}\hat{0}_0(\hat{t},\hat{r})}{\hat{\partial}\,\hat{r}^k}\,\hat{}^A. \qquad (2.113)$$

The reader should keep in mind the "direct universality" of isoequations (2.66) for all possible, well behaved, nonpotential and integro-differential systems, which is evidently important for studies in the interior problem of interest for this monograph.

The *isodual Lagrange-Santilli equations* for the characterization of antimatter in vacuum are defined on structures (2.59), (2.60) and are given by [loc. cit.]

$$L_k{}^d(P_o{}^d) = \left\{ \frac{d^d}{d^d t^d} \frac{\partial^d L^d(t^d, r^d, v^d)}{\partial^d v^{kd}} - \frac{\partial^d L^d(t^d, r^d, v^d)}{\partial^d r^{kd}} \right\}(P_o{}^d) = 0, \qquad (2.114)$$

and they provide a direct analytic representation of the isodual newton-Santilli

equations (2.61).

The *isodual Lagrange–Santilli isoequations* for the characterization of antimatter in interior conditions are defined on structures (2.62), (2.63) and are given by [loc. cit.]

$$\hat{L}_k^{d}(\hat{P}_o^{\,d}) = \Big\{ \frac{\hat\partial^d}{\hat\partial^d \hat t^d} \frac{\hat\partial^d \hat{L}^d(\hat t^d, \hat r^d, \hat v^d)}{\hat\partial^d \hat v^{kd}} - \frac{\hat\partial^d \hat{L}^d(\hat t^d, \hat r^d, \hat v^d)}{\hat\partial^d \hat r^{kd}} \Big\} (\hat{P}_o^{\,d}) = 0\,, \quad (2.115)$$

It is evident that the latter equations provide a direct analytic representation of the isodual Newton–Santilli isoequations (2.64) for antimatter in interior conditions, and they contain as particular case the *isodual Lagrange equations*.

II.2.11. Isotopies and isodualities of Hamiltonian mechanics.

The Class I isotopies of the Legendre transform based on the isodifferential calculus were introduced for the first time in memoir [100]. They are defined on the isospace

$$\hat S(\hat t, \hat x, \hat p) = \hat E(\hat t) \times \hat E(\hat r, \hat\delta, \hat R) \times \hat E(\hat p, \hat\delta, \hat R)\,, \quad (2.116)$$

with total isounits

$$\hat 1_{tot} = \hat 1_t \times \hat 1_x \times \hat 1_p = \hat 1_t \times \hat 1 \times \hat T, \quad (2.117)$$

where *momentum isounit* $\hat 1_p$ is assumed to be equal to the space isotopic element $\hat T_x$ because of the covariant character of $\hat p$.

It should be indicated that, in view of the independence of the variables $\hat p_k$ from $\hat r^k$, we can introduce a new isounit $\hat W = \hat Z^{-1}$ for the isospace $\hat E(\hat p, \hat\delta, \hat R)$ which is different than the unit $\hat 1 = \hat T^{-1}$ of isospace $\hat E(\hat r, \hat\delta, \hat R)$, in which case the total unit is $\hat 1_2 = \hat 1_t \times \hat 1 \times \hat W$. Selection (A.10) is based on the simplest possible case $\hat W \equiv \hat 1$ which is recommendable from the geometric isotopies of Part 4. Other alternatives belong the the problem of the degrees of freedom of the isotopic theories which is not studied in this paper for brevity.

We therefore have the following isodifferentials and isoderivatives

$$\hat\partial \hat t = \hat 1_t \times d\hat t\,, \quad \hat\partial \hat r^k = \hat 1^k{}_i \times d\hat r^i\,, \quad \hat\partial \hat r^i \hat / \hat\partial \hat r^j = \delta^i{}_j \times \hat 1\,, \quad \text{etc.}\,,$$

$$\hat\partial \hat p_k = \hat T_k{}^i \times d\hat p_i\,, \qquad \hat\partial \hat p^k = \hat 1^k{}_i \times d\hat p^i, \qquad \hat\partial \hat p_i \hat / \hat\partial \hat p_j = \delta_i \times \hat 1^j\,, \quad \text{etc.}\,, \quad (2.118)$$

The *isocanonical momentum* is then characterized by [100]

$$\hat{p}_k = \frac{\partial \hat{L}(\hat{t}, \hat{r}, \hat{v})}{\partial \hat{v}^k} = \hat{m} \hat{\times} \hat{v}_k - \hat{U}_k(\hat{t}, \hat{r}), \tag{2.119}$$

under the following regularity condition in a (2n+1)-dimensional region $\hat{D}$ of isopoints $(\hat{t}, \hat{r}, \hat{p})$

$$\text{Det.}\left(\frac{\partial^2 \hat{L}(\hat{t}, \hat{r}, \hat{v})}{\partial \hat{v}^i \partial \hat{v}^j} \right) (\hat{D}) \neq 0 . \tag{2.120}$$

thus admitting a unique set of implicit isofunctions $\hat{v}^k = \hat{f}^k(\hat{t}, \hat{r}, \hat{p})$. The *isolegendre transform* can then be defined by [5g]

$$\hat{L}(\hat{t}, \hat{r}, \hat{v}(\hat{t}, \hat{r}, \hat{p}) = \hat{p}_k\, \hat{v}^k(\hat{t}, \hat{r}, \hat{p}) - \hat{\tfrac{1}{2}} \hat{\times} \hat{m} \hat{\times} \hat{v}_i(\hat{t}, \hat{r}, \hat{p})\; \hat{v}^i(\hat{t}, \hat{r}, \hat{p}) + \tag{2.121}$$

$$+ \hat{U}_k(\hat{t}, \hat{r}) \hat{\times} \hat{v}^k(\hat{t}, \hat{r}, \hat{p}) + \hat{U}_o(\hat{t}, \hat{r}) = \hat{p}_k \hat{\times} \hat{p}^k \hat{/} \hat{2} \hat{\times} \hat{m} + \hat{V}^k(\hat{t}, \hat{r}) \hat{\times} \hat{p}_k + \hat{V}^o(\hat{t}, \hat{r}) = \hat{H}(\hat{t}, \hat{r}, \hat{p}).$$

By using the unified notations

$$\hat{b} = \{ \hat{b}^\mu \} = \{ \hat{r}^k, \hat{p}_k \}, \quad \hat{c}^\mu = \hat{d}\hat{b}^\mu \hat{/} \hat{d}\hat{t}, \tag{2.122}$$

$$\hat{R}^\circ = \{ \hat{R}^\circ{}_\mu \} = \{ \hat{p}_k, \hat{0} \}, \quad \mu = 1, 2, ..., 2n, \quad k = 1, 2, ..., n, \tag{2.123}$$

the isotopies of the celebrated Hamilton principle lead to the following:

Theorem 2.6 (Hamilton-Santilli Necessary Condition) [100]: *A necessary condition for an isofunctional in isocanonical form whose integrand is sufficiently smooth and regular in a region of points* $(\hat{t}, \hat{b}, \hat{c})$

$$\hat{A} = \hat{\int}_{\hat{t}_1}^{\hat{t}_2} \hat{d}\hat{t}\, (\hat{p}_k \hat{\times} \hat{d}\hat{r}^k / \hat{d}\hat{t} - \hat{H})\, (\hat{P}_o) = \hat{\int}_{\hat{t}_1}^{\hat{t}_2} \hat{d}\hat{t}\, (\hat{R}^\circ_\mu \hat{\times} \hat{c}^\mu - \hat{H})\, (\hat{P}_o) \tag{2.124}$$

to have an extremum along an isopath $\hat{P}_o$ *is that all the following "Hamilton-santilli isoequations" hold in the disjoint or unified notations*

$$\frac{\hat{d}\hat{r}^k}{\hat{d}\hat{t}} = \frac{\hat{\partial} \hat{H}(\hat{t}, \hat{r}, \hat{p})}{\hat{\partial} \hat{p}_k}, \qquad \frac{\hat{d}\hat{p}_k}{\hat{d}\hat{t}} = -\frac{\hat{\partial} \hat{H}(\hat{t}, \hat{r}, \hat{p})}{\hat{\partial} \hat{r}^k}, \tag{2.125a}$$

$$\hat{\omega}_{\mu\nu} \hat{\times} \frac{\partial \hat{b}^{\nu}}{\hat{\partial t}} = \frac{\partial \hat{H}(\hat{t}, \hat{b})}{\partial \hat{b}^{\mu}}, \tag{2.125b}$$

$$\frac{\partial \hat{b}^{\mu}}{\hat{\partial t}} = \hat{\omega}^{\mu\nu} \hat{\times} \frac{\partial \hat{H}(\hat{t}, \hat{b})}{\partial \hat{b}^{\nu}}, \tag{2.125c}$$

$$\hat{\omega}_{\mu\nu} = \omega_{\mu\nu} \hat{\times} \hat{1}, \quad \hat{\omega}^{\mu\nu} = \omega^{\mu\nu} \times \hat{1}, \tag{2.125d}$$

$$(\omega_{\mu\nu}) = \left(\frac{\partial R^{\circ}_{\nu}}{\partial \hat{b}^{\mu}} - \frac{\partial R^{\circ}_{\mu}}{\partial \hat{b}^{\nu}} \right) = \begin{pmatrix} 0_{N\times N} & -1_{N\times N} \\ 1_{N\times N} & 0_{N\times N} \end{pmatrix}, \tag{2.125e}$$

$$(\omega^{\alpha\beta}) = \left(\frac{\partial R^{\circ}_{\nu}}{\partial \hat{b}^{\mu}} - \frac{\partial R^{\circ}_{\mu}}{\partial \hat{b}^{\nu}} \right)^{-1} = \begin{pmatrix} 0_{N\times N} & 1_{N\times N} \\ -1_{N\times N} & 0_{N\times N} \end{pmatrix}, \tag{2.125f}$$

with integrated form

$$\hat{b}(t) = \{ \hat{e}^{\,-\hat{t} \hat{\times} \partial_{\mu} \hat{H} \hat{\times} \hat{\omega}^{\mu\nu} \partial_{\nu}} \} \times \hat{b}(\hat{0}) . \tag{2.126}$$

Note that the latter quantities are the *conventional* covariant and contravariant canonical tensors, respectively, which hold in view of the identities originating from properties (2.72) and values (2.76)

$$\partial \hat{R}^{\circ}_{\nu} / \partial \hat{b}^{\mu} \equiv \partial R^{\circ}_{\nu} / \partial b^{\mu}. \tag{2.128}$$

The equivalence of Santilli's isolagrangian and isohamiltonian equations under the assumed regularity and invertibility of the isolegendre transform can be proved as in the conventional case (see, e.g., Ref. 109] Sect. 3.8).

The novel brackets characterized by the isohamilton's equations between two isofunctions $\hat{A}(\hat{b})$, $\hat{B}(\hat{b})$ on the phase isospace, first achieved in Ref. [100], can be written

$$[\hat{A} \hat{,} \hat{B}] = \frac{\partial \hat{A}}{\partial \hat{r}^{k}} \hat{\times} \frac{\partial \hat{B}}{\partial \hat{p}_{k}} - \frac{\partial \hat{B}}{\partial \hat{r}^{k}} \hat{\times} \frac{\partial \hat{A}}{\partial \hat{p}_{k}} = \left(\frac{\partial A}{\partial r^{k}} \frac{\partial B}{\partial p_{k}} - \frac{\partial B}{\partial r^{k}} \frac{\partial A}{\partial p_{k}} \right) \times \hat{1}, \tag{2.128}$$

and they *formally* coincide with the conventional Poisson brackets. The verification of Lie's axioms by the above brackets on isospace over isofields is

evident. The isobrackets then provide the *fundamental classical realization of the Lie-Santilli isoalgebras* (see Part 3).

Note that the projection of the above brackets in conventional space over conventional fields is given by

$$[A, B]_{Isotopic} = \frac{\partial A}{\partial r_i} \hat{T}_i^k(t, r, p, \ldots)\, \delta_{kj} \frac{\partial B}{\partial p_j} - \frac{\partial B}{\partial r_i} \hat{T}_i^k(t, r, p, \ldots)\, \delta_{kj} \frac{\partial A}{\partial p_j} \tag{2.129}$$

do not verify the Lie axioms (because they generally violate the Jacobi law). This illustrates again, this time from an algebraic profiles, the structural differences of the isotopic theory in its dual formulation, that on isospace over isofields and that projected on conventional spaces over conventional fields.

Brackets (2.81) can be written in unified notation (2.76)

$$[\hat{A}, \hat{B}] = \frac{\partial \hat{A}}{\partial \hat{b}^\mu} \hat{\times} \hat{\omega}^{\mu\nu} \hat{\times} \frac{\partial \hat{B}}{\partial \hat{b}^\nu} = \frac{\partial A}{\partial b^\alpha} \times {}_\mu{}^\alpha \times \omega^{\mu\nu} \times \hat{T}_\nu{}^\beta \frac{\partial B}{\partial \hat{b}^\beta} = \frac{\partial A}{\partial \hat{b}^\alpha} \times \omega^{\alpha\beta} \times \frac{\partial B}{\partial \hat{b}^\beta} \tag{2.130}$$

where , again, all contractions are on isospace and the last identity occurs in view of the properties for diagonal isounits

$$\hat{T}_\mu{}^\alpha\, \omega^{\mu\nu}\, \hat{T}_\nu{}^\beta = \omega^{\alpha\beta}. \tag{2.131}$$

It is also possible to show that isohamiltonian equations in their isoexponentiated form provides a classical realization of the *Lie-Santilli isogroups* (see Part 3).

The *isotopic Hamilton-Jacobi equations*, identified for the first time by Santilli via the isodifferential calculus in memoir [100], are given by

$$\frac{\partial \hat{A}}{\partial \hat{t}} + \hat{H}(t, \hat{x}, \hat{p}) = 0, \quad \frac{\partial \hat{A}}{\partial \hat{r}^k} - \hat{p}_k = 0, \quad \frac{\partial \hat{A}}{\partial \hat{p}_k} = 0, \tag{2.132}$$

plus initial conditions $\partial\hat{A}/\partial\hat{x}^{\circ k} = \hat{p}_k{}^\circ$, where $\hat{x}^\circ$ and $\hat{p}^\circ$ are iso constants, and they are at the foundation of the *isoquantization* of the theory studied later on.

An important property of the Hamilton-Santilli isomechanics is that it is as invariant as the conventional Hamiltonian mechanics. In particular, the *isocanonical isotransforms* are defined via the invariance of the Lie-Santilli

isotensor under the transforms $\hat{b} \rightarrow \hat{b}'(\hat{b})$

$$\hat{\omega}^{\mu\nu} = \mu\nu \times \hat{1} = \frac{\partial \hat{b}^{\mu}}{\partial \hat{b}^{\alpha}} \hat{\times} \hat{\omega}^{\alpha\beta} \hat{\div} \frac{\partial \hat{b}'^{\nu}}{\partial \hat{b}^{\beta}} \equiv \hat{\omega}^{\mu\nu}. \tag{2.133}$$

The study in details of the above isotransformation theory is suggested to the reader interested in learning these new methods.

Note the *abstract identity between the conventional and isotopic mechanics*. Since the isounits are positive–definite, at the abstract level there is no distinction between dt and $\hat{d}\hat{t}$, dx and $\hat{d}\hat{x}$, etc. Therefore, the isonewton, isolagrange and isohamilton equations coincide at the abstract level with the conventional equations. This illustrates again the axiom–preserving character of the isotopies.

Note the *direct universality* of Santilli's isohamiltonian mechanics in the fixed inertial frame of the observer, which should be compared with the corresponding *lack* of universality of the conventional Hamiltonian mechanics. This direct universality permits an alternative formulation of Darboux's theorem of the symplectic geometry outlined in the next appendix, according to which, under the needed smoothness and regularity conditions, all possible locally *nonhamiltonian* systems can always be locally *isohamiltonian*.

The *isodual Hamilton–Santilli equations* are defined on structures

$$S^d(t^d,x^d,p^d) = E^d(t^d,R^d_t) \times E^d(x^d,\delta^d,R^d) \times E^d(p^d,\delta^d,R^d), \tag{2.134}$$

with *isodual unit*

$$I^d_{tot} = I^d_t \times I^d \times I^d, \qquad I_t^d = -1, \qquad I^d = \mathrm{diag.}\,(-1,-1,-1), \tag{2.135}$$

can be written

$$m^d x^d \frac{d^d v_k{}^d}{d^d t^d} - \frac{d^d}{d^d t^d} \frac{\partial^d U^d(t^d, x^d, v^d)}{\partial^d v^{kd}} + \frac{\partial^d U^d(t^d, x^d, v^d)}{\partial^d x^{kd}} = 0. \tag{2.136}$$

where $H^d = -H$, as the reader is encouraged to verify. The above equations provide an isocanonical representation of the isodual Newton–Santilli isodual equations (2.107).

The *isodual hamilton–Santilli isoequations* are defined don the *isodual isospace*

$$\hat{S}^d(\hat{t}^d,\hat{x}^d,\hat{p}^d) = \hat{E}^d(\hat{t}^d,\hat{R}^d{}_t)\times\hat{E}^d(\hat{x}^d,\hat{\delta}^d,\hat{R}^d)\times\hat{E}^d(\hat{p}^d,\hat{\delta}^d,\hat{R}^d), \qquad (2.137)$$

with *isodual isounit*

$$\hat{1}^d{}_{tot} = \hat{1}^d{}_t \times \hat{1}_x{}^d \times \hat{1}_p{}^d = \hat{1}_t{}^d \times \hat{1}^d \times \hat{T}^d, \qquad \hat{1}_t{}^d = -\hat{1}_t, \quad \hat{1}^d = -\hat{1} = (\hat{T}^{d-1}, \qquad (2.138)$$

can be written

$$\hat{\omega}_{\mu\nu}{}^d \times^d \frac{\hat{\partial}^d \hat{b}^\nu}{\hat{\partial}^d \hat{t}^d} = \frac{\hat{\partial}^d \hat{H}^d(\hat{t}^d, \hat{b}^d)}{\hat{\partial}^d \hat{b}^{\mu d}}, \qquad (2.139)$$

where again $\hat{H}^d = -\hat{H}$, and they evidently represent the isodual isonewton equations (2.110).

The *isodual Hamilton-Jacobi-Santilli equations* [100] are evidently given by

$$A^d = \int_{t_1{}^d}^{t_2{}^d} (p^d{}_k \times^d d^d r^{dk} - H^d \times^d d^d t^d) = -A, \qquad (2.140a)$$

$$\frac{\partial^d A^d}{\partial^d t^d} + H^d(t^d, x^d, p^d) = 0, \qquad \frac{\partial^d A^d}{\partial^d r^{dk}} - p^d{}_k = 0, \qquad \frac{\partial^d A^d}{\partial^d p^d{}_k} = 0, \qquad (2.140b)$$

and they are fundamental for the characterization of the isodual image of quantum mechanics, while the *isodual Hamilton-Jacobi-santilli isoequations* can be written [loc. cit.]

$$\hat{A}^d = \int_{\hat{t}_1{}^d}^{\hat{t}_2{}^d} \hat{p}^d{}_k \hat{\times}^d \hat{\partial}^d \hat{r}^{dk} - \hat{H}^d \hat{\times}^d \hat{\partial}^d \hat{t}^d) = -\hat{A}, \qquad (2.141a)$$

$$\frac{\hat{\partial}^d \hat{A}^d}{\hat{\partial}^d \hat{t}^d} + \hat{H}^d(\hat{t}^d, \hat{x}^d, \hat{p}^d) = 0, \qquad \frac{\hat{\partial}^d \hat{A}^d}{\hat{\partial}^d \hat{r}^{dk}} - \hat{p}^d{}_k = 0, \qquad \frac{\hat{\partial}^d \hat{A}^d}{\hat{\partial}^d \hat{p}^d{}_k} = 0, \qquad (2.141b)$$

Note *the complete equivalence in the treatment of matter and antimatter beginning at the primitive Newtonian level which then persists at all subsequent level including quantization, quantum field theory and electroweak interactions* [100,101].

For completeness, we mention that Santilli had already achieved in monograph [110] the direct universality for the representation of nonconservative Newtonian systems via the use of *Birkhoffian mechanics*, but under the condition

that they are (well behaved and) *local-differential* (as necessary from the use of the conventional symplectic geometry).

In memoir [100] Santilli first, extended the above direct universality to include nonlocal-integral systems; and, second, he reduced the representation to a form admitting the *conventional* canonical structure ω.

Both results are important for this volume. In fact, the representation of generally nonlocal-integral systems is essential for an effective study of interior dynamical systems at both classical and operator levels, as we shall see. Moreover, the quantization of Birkhoffian mechanics yielded no meaningful operator mechanics. The latter impasse was resolved precisely in memoir [100] via the isohamiltonian mechanics.

The connection between Birkhoff's equations and the Hamilton-Santilli isoequations is intriguing. Recall that the former are characterized by the most general possible, nowhere degenerate, exact symplectic structure

$$\Omega_{\mu\nu}(b) = \partial R_\nu / \partial b^\mu - \partial R_\mu / \partial b^\nu , \tag{2.142}$$

and can be written

$$\Omega_{\mu\nu}(b) \times \frac{db^\nu}{dt} = \frac{\partial H(t, b)}{\partial b^\mu} . \tag{2.143}$$

It is then easy to see that the above equations are reducible to the Hamilton-Santilli isoequations via the factorization of the general symplectic structure into the conventional symplectic structure and a factor interpreted as the 2N-dimensional isounit [100]

$$\Omega_{\mu\nu}(b) = \hat{1}^\alpha{}_\mu \times \omega_{\alpha\nu} . \tag{2.144}$$

In fact, under the latter decomposition, Eq.s (2.93) reduce to

$$\omega_{\mu\nu} \times \frac{db^\nu}{dt} = \hat{T}_\mu{}^\alpha \times \frac{\partial H(t, b)}{\partial b^\alpha} = \frac{\partial H(t, b)}{\partial b^\mu} . \tag{2.145}$$

namely, they reduce to Eq.s (2.78b) for $\hat{1}_t = I$.

The reader should keep in mind that the direct universality of the Hamilton-Santilli isoequations establishes the corresponding direct universality of the Lie-Santilli isotheory in classical mechanics with a corresponding direct

universality for operator formulations indicated later on.

II.2.12: Isotopies and isodualities of quantization

As it is well known, quantum mechanics provides the fundamental operator realization of Lie's theory. It is therefore important to outline the operator realization of the Lie-Santilli isotheory which was first proposed in ref. [53] under the name of *hadronic mechanics* and then studied in numerous subsequent papers (see ref.s [116] for a comprehensive presentation), but reached maturity of formulation only in the recent memoir [101] following the appearance of the isodifferential calculus in the preceding memoir [100].

The reader may be interested in knowing that hadronic mechanics was specifically built for the problems of structure and interactions of strongly interacting particles generically called hadrons. Recall that quantum mechanics is strictly local-differential and potential-Hamiltonian. As such, it has resulted to be *exactly valid* for electromagnetic and weak interactions, although there are historical doubts whether the same discipline can also be *exact* for strong interactions.

This is due to the fact that the range of the strong interactions coincides with the size (charge distribution) of all hadrons. As a result, a necessary condition to activate the strong interactions is that the hyperdense hadrons enter into conditions of mutual penetration-overlapping, thus resulting in the most general known systems which are nonlinear in the wavefunctions (and possibly their derivatives), nonlocal-integral (over the volume of overlapping) and nonpotential-nonhamiltonian and, therefore, nonunitary (because of the contact interactions which are absent in the electroweak interactions). In turn, any nonlinear, nonlocal and nonunitary study of strong interactions requires a structural revision of quantum mechanics beginning with its topology.

The understanding is that the *approximate validity* of quantum mechanics for strong interactions is unquestionable. We are therefore referring to *corrections* of the quantum descriptions due to internal nonlinear, nonlocal and nonunitary effects.

Santilli [53] proposed the construction of the *isotopies of quantum mechanics* under the name of hadronic mechanics precisely for the treatment of the latter contributions in a form which preserves the original quantum mechanical axioms.

The fundamental dynamical equations of hadronic mechanics can be uniquely and unambiguously derived from the Hamilton-Santilli isomechanics via

the isotopies and isodualities of conventional naive or symplectic quantization.

Recall that the *naive quantization* can be expressed via the mapping (for $\hbar$ = 1)

$$A = \int_{t_1}^{t_2} (p_k dx^k - H\, dt) \rightarrow -i \times \hbar \times \mathrm{Ln}\, \psi(t, x) = -i \times \mathrm{Ln}\, \psi(t, x) \quad (2.146)$$

under which the conventional Hamilton-Jacobi equations are mapped into the Schrödinger's equations,

$$\partial_t A + H = 0 \rightarrow i \times \partial_t \psi = H \times \psi,$$

$$\partial_k A - p_k = 0 \rightarrow -i \times \partial_k \psi = p_k \times \psi. \quad (2.147)$$

By recalling the *seven* classes of new mathematics (Sect. 2.1), the above maps has been subjected to *seven* different liftings each one characterizing a novel mechanics. By including the conventional quantum mechanics as a trivial particular case for $\hat{1}$ $\hbar$, the terms "hadronic mechanics" are today referred to *eight* branches, the conventional, isotopic, genotopic and hyperstructural mechanics, plus their four isoduals.

The first four formulations all obey the same abstract axioms, by conception and construction. For this reason, santilli insists that the isotopic, genotopic and hyperstructural mechanics *do not* constitute "new mechanics" , but merely "new realizations" of the abstract axioms of quantum mechanics. In particular, they constitute a form of "completion" of quantum mechanics much along the historical argument by Einstein, Podolsky and Rosen, as indicated beginning from the title of memoir [101]. The above mechanics are used for the sole characterization of *particles*. The isotopic, genotopic and hyperstructural branches have also provided a novel operator realization of gravity for matter for physical conditions of increasing complexity.

The remaining four branches (isodual quantum, isodual isotopic, isodual genotopic and isodual hyperstructural mechanics) are anti-isomorphic to the preceding ones. As such, they have resulted to provide an intriguing novel characterization of *antiparticles* [101,105,106]. the latter branches have also provided an intriguing and novel operator form of gravity for antimatter with conditions of increasing complexity. In this section we can only review only the following three new quantization.

1) ***Naive isodual quantization.*** It is characterized the mapping of the isodual action (2.140a)

$$A^d = \int^d {}_{t_1{}^d}^{t_2{}^d} (p^d{}_k \times^d d^d r^{dk} - H^d \times^d d^d t^d) \rightarrow -i^d \times^d \hbar^d \, Ln^d \psi^d (t^d, x)^d. \quad (2.148)$$

where one should remember that $i^d = -\bar{i} = i$, $\hbar^d = -\hbar^\dagger = -\hbar$, $Ln^d\psi^d = - Ln\psi^d$, $\psi^d = -\psi^\dagger$, under which the isodual equations (2.140b) are mapped into the operator form,

$$\frac{\partial^d A^d}{\partial^d t^d} + H^d(t^d, x^d, p^d) = 0 \quad \rightarrow \quad i^d \times^d \partial_t{}^d \psi^d = H^d \times^d \psi^d ,$$

$$\frac{\partial^d A^d}{\partial^d r^{dk}} - p^d{}_k = 0 \quad \rightarrow \quad - i^d \times^d \partial^d{}_k \psi^d = p^d{}_k \times^d \psi^d . \quad (2.149)$$

which characterize the ***isodual quantum branch of hadronic mechanics.*** The latter is a novel, hitherto unknown, anti-isomorphic image of quantum mechanics identified by Santilli [101,105,106] for the characterization of antiparticles in vacuum.

Its fundamental assumption is the lifting of the unit of quantum mechanics, Planck's constant, under isoduality,

$$\hbar = I \quad \rightarrow \quad I^d = - I , \quad (2.150)$$

and the reconstruction of the entire theory in such a way to admit the new unit I^d as the correct, left and right unit at all levels. This implies the change of the sign of all characteristics of particles, that is, not only of the charge, but also of masse, energy, time etc., in a way fully analogy with the classical Newtonian counterpart of Sect. 2.8.

The reader should be aware that the above new quantization resolves the following historical shortcoming of contemporary theoretical physics. Recall that, prior to Santilli's studies, heretical physics used *only one quantization*. as a result, the operator image of contemporary classical descriptions of antiparticles is not the correct charge conjugate, but instead the state of an ordinary particles with the mere change of the sign of the charge.

By constructing a new theory for antiparticles which begins at the Newtonian level and then continues with its own quantization, Santilli has resolved this historical impasse. In fact, at the operator level isoduality has resulted to be equivalent to charge conjugation, as we shall review later on.

2) ***Naive isoquantization.*** It is characterized by the map of the isoaction

(2.124)

$$\hat{A} = \int_{t_1}^{t_2} [\, \hat{p}_k \hat{\times} \hat{d}\hat{x}^k - \hat{H} \hat{\times} \hat{d}\hat{t}] \quad \to \quad -\hat{\imath} \hat{\times} \text{Isoln}\, \hat{\psi}(\hat{t}, \hat{x}) = -\, i \times \hat{1}\, \text{Ln}\, \hat{\psi}(\hat{t}, \hat{x}) \quad (2.151)$$

where $\hat{\imath} = i \times \hat{1}$ and we have used the notion of *isologarithm*, $\text{Isoln}\, \hat{\psi} = \hat{1} \times \text{Ln}\, \hat{\psi}$. This implies the map of the Hamilton–Jacob–Santilli isoequations (2.132) into the operator forms

$$\partial_t \hat{A} + \hat{H} = 0 \quad \to \quad \hat{\imath} \hat{\times} \partial_t \hat{\psi} = \hat{H} \hat{\times} \hat{\psi} = \hat{H} \times \hat{T} \times \hat{\psi},$$

$$\partial_k \hat{A} - \hat{p}_k = 0 \quad \to \quad -\hat{\imath} \hat{\times} \partial_k \hat{\psi} = \hat{p}_k \hat{\times} \hat{\psi} = \hat{p}_k \times \hat{T} \times \hat{\psi}, \qquad (2.152)$$

which are valid when $\hat{1}$ is independent from space and time coordinates, otherwise $\hat{H}$ is replaced by the more general operator $\hat{H}^{Eff}$ inclusive of the latter effects [128].

The above equations have the desired manifest isotopic structure first discovered by Mignani and, independently, by Myung and Santilli (see Ref. [116] for details and literature). They constitute the fundamental isoschrödinger's equations of the ***isotopic branch of of hadronic mechanics*** which is used for the characterization of particles in interior conditions, operator gravity of particles and other applications.

The rigorous operator map is given by the *isotopies of symplectic quantization* [116] and yields exactly the same results.

As one can see, the fundamental assumption of isoquantization is the lifting of the basic unit of quantum mechanics, Planck's unit, into a matrix with nonlinear, integro-differential elements

$$\hbar = 1 \quad \to \quad \hat{1} = \hat{1}(t, x, \dot{x}, \psi, \partial\psi, \ldots) > 0, \qquad (2.153)$$

and reconstruction of the formulation in such a way to admit $\hat{1}$ as the correct left and right unit at all levels. Nevertheless, as we shall see shortly, the original value $\hbar$ is recovered identically from $\hat{1}$ under the applicable expectation values. this will confirm the "hidden" character of the isotopies.

2) ***Naive isodual isoquantization.*** It is characterized by the map of the isodual isoaction (2.141a)

$$\hat{A}^d = \int_{\hat{t}_1{}^d}^{d\ \hat{t}_2{}^d} (\hat{p}^d{}_k \hat{\times}^d \hat{d}^d \hat{r}^{dk} - \hat{H}^d \hat{\times}^d \hat{d}\,\hat{t}^d) \to -\hat{\imath}^d \hat{\times}^d \text{Isoln}^d\, \hat{\psi}^d(\hat{t}^d, \hat{x}^d) =$$

$$= -\, i \times \hat{1}^d \times \text{Ln}\, \hat{\psi}^d(\hat{t}^d, \hat{x}^d), \qquad (2.154)$$

under which the isodual Hamilton–Jacobi–Santilli isoequations (2.141b) are mapped into the operator forms

$$\partial^d{}_t \hat{A}^d + \hat{H}^d = 0 \quad \rightarrow \quad \hat{\imath} \,\hat{\times}\, \partial^d{}_t \hat{\psi}^d = \hat{H}^d \times^d \hat{\psi}^d = \hat{H}^d \times \hat{T}^d \times \hat{\psi}^d ,$$

$$\partial^d{}_k \hat{A}^d - \hat{p}^d{}_k = 0 \quad \rightarrow \quad -\hat{\imath}^d \times^d \partial^d{}_k \hat{\psi}^d = \hat{p}^d{}_k \times^d \hat{\psi}^d = \hat{p}^d{}_k \times \hat{T}^d \times \hat{\psi}^d , \quad (2.155)$$

which characterize the ***isodual isotopic branch of hadronic mechanics,*** and it is used for the characterization of antiparticles in interior conditions, operator gravity for antimatter, and other applications.

This third branch is evidently characterized by the lifting of Planck's constant

$$\hbar = 1 \quad \rightarrow \quad \hat{1}^d = \hat{1}^d(t, x, \dot{x}, \psi, \partial\psi, \dots) < 0 , \qquad (2.156)$$

under which all characteristics of the isodual branch change sign, and the reconstruction of the mechanics to admit $\hat{1}^d$ as the left and right unit at all levels.

II.2.13. Isotopies and isodualities of quantum mechanics.

As indicated earlier, hadronic mechanics has reached maturity of formulation only in the recent (104 pages long) memoir [101]. To avoid a prohibitive length, in this section we shall outline the mathematical structure of the three branches of hadronic mechanics under consideration here. For clarity, only *nonrelativistic hadronic mechanics* will be considered in this section. relativistic extensions will be indicated later on.

Let ξ be the enveloping associative operator algebra of quantum mechanics with elements A, B, ..., unit $\hbar = I$ and conventional associative product $A \times B = AB$ over the fields of complex numbers $C = C(c,+,\times)$, and let $\mathcal{H}$ be a conventional Hilbert space with states $|\psi\rangle, |\phi\rangle, \dots$ and inner product $\langle \psi | \phi \rangle = \int d^3x \psi^\dagger(t,x)\phi(t, x)$ over C.

I) ***Mathematical structure of the isodual quantum branch.*** It is given by the *isodual enveloping associative algebra* ξ^d with elements $A^d = -A^\dagger$, $B^d = -B^\dagger$, isodual unit $\hat{1}^d = -I$, and isodual product

$$A^d \times^d B^d = A^d \times I^d \times B^d = -(A^\dagger \times B^\dagger) , \qquad (2.157)$$

over the isodual field $C^d = C^d(c^d,+,\times^d)$, $c^d = -\bar{c}$, equipped with the *isodual Hilbert space* $\mathcal{H}^d$ with isodual states $|\psi>^d = -|\psi>^\dagger$, and isodual inner product over C^d

$$<\phi|\times|\psi>^d = <\phi|^d \times I^d \times |\psi>^d \times I^d \in C^d. \tag{2.158}$$

We then have the *isodual eigenvalue equations*

$$H^d \times^d |\psi>^d = E^d \times^d |\psi>^d, \tag{2.159}$$

characterizing *negative energies* $E^d = -E < 0$, as desired.

The *isodual eigenvalues* are then given by

$$<^d H^d>^d = <\psi|^d \times^d H^d \times^d |\psi>^d /^d <\psi|^d \times^d |\psi>^d = E^d = -E, \tag{2.160}$$

thus recovering the isodual eigenvalues, as needed for consistency.

The above isodual theory stems from a novel invariance, the *isoselfduality of the normalization of the Hilbert space* [101,105] namely, its invariance under isoduality (Sect. 2.3)

$$<\psi|\times|\psi>\times I \equiv <\phi|^d \times I^d \times |\psi>^d \times I^d, \tag{2.161}$$

which assures that all physical laws of particles also holds for antiparticles.

For remaining aspects of the isodual branch of hadronic mechanics we refer the reader to Ref.s [101,105].

Note that invariance (2.161) has remaining undetected throughout this century. This should not be surprising because its identification requires the prior discovery of *new numbers,* those with *negative units.*

2) ***Mathematical structure of the isotopic branch.*** It is characterized by

1) the Class I lifting of the (space) unit $I \to \hat{I} = \hat{T}^{-1} > 0$ with consequential isofields of real $\hat{R} = \hat{R}(\hat{n},+,\hat{\times})$ and complex isonumbers $\hat{C}$ $\hat{C}(\hat{n},+,\hat{\times})$;

2) The corresponding lifting of the quantum mechanical representation spaces, such as the Euclidean $E(x,\delta,R)$ spaces into their isotopic form $\hat{E}(\hat{x},\hat{\delta},\hat{R})$;

3) The lifting of the enveloping operator algebras ξ into the *enveloping isoassociative algebra* $\hat{\xi}$ with the same original elements $\hat{A} = A$, $\hat{B} = B$, ..., only written in isospace, now equipped with the isounit $\hat{I}$ and the isoassociative product

$$\hat{A} \hat{\times} \hat{B} = \hat{A} \times \hat{T} \times \hat{B}, \tag{2.162}$$

as well as the lifting of the Hilbert space $\mathcal{H}$ into the *isohilbert space* $\hat{\mathcal{H}}$ with

isostates $|\hat{\psi}>$, $|\hat{\phi}>$, ... and *isoinner product*, first identified by Myung and Santilli (see ref. [116] for details and literature)

$$\hat{\mathcal{H}}: <\hat{\phi}\hat{\uparrow}\hat{\psi}> = <\hat{\phi}|\times\hat{T}\times|\hat{\psi}>\times\hat{1} \in \hat{C}(\hat{c},+,\hat{\times}). \quad (2.162)$$

The main equations of hadronic mechanics defined in terms of the above mathematical methods are explicitly given by: *the isoschrödinger equations for the linear momentum*

$$-\,i\times\hat{\partial}_k\hat{\psi}(t,\hat{x}) = -\,i\times T_k{}^1\,\partial_1\,\psi(t,r) = \hat{p}_k\,\hat{\times}\,\psi(t,x) = \hat{p}_k\times\hat{T}\times\hat{\psi}(t,\hat{x}), \quad (2.163)$$

and the related *fundamental isocommutation rules*

$$[\,\hat{p}_i\,\hat{,}\,\hat{x}^j\,] = \hat{p}_i\,\hat{\times}\,\hat{x}^j - \hat{x}^j\,\hat{\times}\,\hat{p}_i = -\,\delta_i{}^j,\ \ [\,\hat{p}_i\,\hat{,}\,\hat{p}_j\,] = [\,\hat{x}^i\,\hat{,}\,\hat{x}^j\,] \equiv 0, \quad (2.164)$$

(where we have used properties (2.46); the *isoscrödinger equation for the energy*

$$i\times\hat{\partial}_t\,\hat{\psi}(t,\hat{x}) = i\,\hat{T}_t\times\partial_t\,\psi(t,x) = \hat{H}\,\hat{\times}\,\hat{\psi}(t,\hat{x}) =$$

$$= \hat{H}\times\hat{T}\times\hat{\psi}(t,\hat{x}) = \hat{E}\,\hat{\times}\,\hat{\psi}(t,\hat{x}) = E\times\hat{\psi}(t,\hat{x}),$$

$$\hat{H} = \hat{H}^{\hat{\dagger}},\ \ \hat{E} = E\times\hat{1} \in \hat{R}(\hat{n},+,\hat{\times}),\ \ E \in R(n,+,\times); \quad (2.165)$$

and the *isoheisenberg equation*

$$i\,\hat{d}\,\hat{Q}\,/\,\hat{d}t = [\,\hat{Q}\,\hat{,}\,\hat{H}] = \hat{Q}\,\hat{\times}\,\hat{H} - \hat{H}\,\hat{\times}\,\hat{Q} = \hat{Q}\times\hat{T}\times\hat{H} - \hat{H}\times\hat{T}\times\hat{Q}. \quad (2.166)$$

with integrated form

$$\hat{Q}(t) = e^{i\times\hat{H}\times\hat{T}\times t}\times\hat{Q}(0)\times e^{-i\times t\times\hat{T}\times\hat{H}}, \quad (2.167)$$

first identified by Santilli in the original proposal to build hadronic mechanics [53] and generally called *Heisenberg-Santilli equations*.

It should be recalled for subsequent needs that *the condition of isohermiticity on an isohilbert space coincides with the conventional Hermiticity,* $\hat{H}^{\hat{\dagger}} = H^{\dagger}$. As a consequence, *all operators which are Hermitean-observable in quantum mechanics remain so in hadronic mechanics.*

For the isotopies of the remaining aspects of quantum mechanics we refer for brevity the interested reader to monograph [116]. We here merely indicate that,

for isounits of Class I, all distinctions between quantum and hadronic mechanics cease to exist at the abstract, realization-free level for which $R \approx \hat{R}$, $C \approx \hat{C}$, $\xi \approx \hat{\xi}$, $E \approx \hat{E}$, $\mathcal{H} \approx \hat{\mathcal{H}}$, etc. This ultimate abstract unity assures the correct axiomatic structure of hadronic mechanics to such an extent that criticisms on its structure may eventually result to be criticisms on quantum mechanics.

The advantages of hadronic over quantum mechanics are similar to those of the Hamilton-Santilli over the conventional Hamiltonian mechanics. In fact, quantum mechanics can only represent (in first quantization) point-like particles under action-at-a-distance interactions. By comparison, hadronic mechanics can represent (in first isoquantization) the actual nonspherical shape of hadrons, their deformations as well as their nonlocal-integral interactions due to mutual penetrations of the hadrons. The possibilities for broader applications in various disciplines are then evident.

3) ***Mathematical structure of the isodual isotopic branch.***

The isodual Hamilton-Santilli isomechanics is mapped via naive isoquantization into the *isodual hadronic mechanics* which is based on: 1) the isodual isofields of isoreals $\hat{R}^d(\hat{n}^d,+,\hat{\times}^d)$ or isocomplex numbers $\hat{C}^d(\hat{c}^d,+,\hat{\times}^d)$ (Sect. 2.B); 2) the *isodual envelope* $\hat{\xi}^d$ with isodual isounit $\hat{1}^d = -\hat{1}$, isodual elements $\hat{A}^d = -A$, $\hat{B}^d = -B$, etc., and isodual product $\hat{A}^d \hat{\times}^d \hat{B}^d = -\hat{A}\hat{T}\hat{B}$; the *isodual isohilbert space* $\hat{\mathcal{H}}^d$ with isodual isostates $|\hat{\psi}>^d = -<\hat{\psi}|$, etc. and isodual isoinner product $<\hat{\phi}|\hat{T}^d|\hat{\psi}>\hat{1}^d$ over $\hat{C}^d$.

In particular, at this operator level, the isodual map has is equivalent to charge conjugation (see [105] for brevity), although with a number of differences. For instance, charge conjugation maps a particle into an antiparticle *in the same carrier space over the same field*, while isoduality maps a particles in a given carrier space over a given field *into a different carrier space over a different field* (the isodual ones); charge conjugation changes the sign of the charge but preserves the sign of energy and time, while isoduality changes the signs of all physical characteristics, although they are now defined over a field of negative-definite norm; etc.

Note that the naive (or symplectic) isoquantization apply for all possible isoaction (2.51). By recalling the direct universality of the Hamilton-Santilli isomechanics, one can therefore see that hadronic mechanics is also directly universal for all possible (well behaved), integro-differential, operator systems which are nonlinear in the wavefunction and its derivatives and nonhamiltonian [116].

In fact, the isoscrödinger's equations can be explicitly written

$$i \times \hat{1}_t(t, \hat{x}, \hat{p}, \ldots) \times \partial_t |\hat{\psi}> \ = \ \hat{H}(\hat{t}, \hat{x}, \hat{p}) \times \hat{T}(t, \hat{x}, \hat{p}, \hat{\psi}, \partial\psi, \partial\partial\psi, \ldots) \times |\hat{\psi}> . \qquad (2.168)$$

The unrestricted functional dependence of the isotopic element then implies the following:

Theorem 2.7 [116]: *Hadronic mechanics is "directly universal", that is, capable of representing all possible, well behaved, nonlinear, nonlocal-integral and nonpotential-nonhamiltonian operator systems (universality), directly in the coordinates of the observer (direct universality).*

This property is remarkable inasmuch as it establishes the direct universality of the Lie-Santilli isotheory also in its operator realization. Note the mechanism in achieving the above direct universality, which is first referred to a well behaved, but otherwise arbitrary nonlinear, nonlocal and nonhermitean operator $\hat{O}(t, \hat{x}, \hat{p}, \hat{\psi}, \partial\psi, \ldots)$. Then the latter operator is decomposed into the product of two Hermitean operators $\hat{O} = \hat{H}\hat{T}$ under the condition that all nonlinear nonlocal and nonhamiltonian terms are embedded in the isotopic element $\hat{T}$. Finally, the underlying methods are reconstructed with respect to the unit $\hat{1} = \hat{T}^{-1}$ so as to reproduce Hermiticity in isohilbert space.

For the isotopies of the remaining aspects of quantum mechanics we refer for brevity the interested reader to monograph [116]. We here merely indicate that, for isounits of Class I, all distinctions between quantum and hadronic mechanics cease to exist at the abstract, realization-free level for which $R \approx \hat{R}$, $C \approx \hat{C}$, $\xi \approx \hat{\xi}$, $E \approx \hat{E}$, $\mathcal{H} \approx \hat{\mathcal{H}}$, etc. This ultimate abstract unity assures the correct axiomatic structure of hadronic mechanics to such an extent that criticisms on its structure may eventually result to be criticisms on quantum mechanics.

As a result, all properties holding for quantum mechanics also hold for hadronic mechanics. For instance, the condition of Hermiticity on $\hat{\mathcal{H}}$ over $\hat{C}$ *coincides* with that on $\mathcal{H}$ over C. Thus, all quantities which are observables in quantum mechanics remain observable for hadronic mechanics.

Finally, we mention that hadronic mechanics: preserves conventional physical laws, such as Heisenberg's uncertainties, Pauli's exclusion principle, etc.; provides a concrete and explicit realization of the theory of "hidden variables"; and ultimately results to be a form of "completion" of quantum mechanics much along the historical teaching of Einstein, Podolsky and Rosen. For all these aspects, we suggest to consult memoir [101] for brevity.

The advantages of hadronic over quantum mechanics are similar to those of the Hamilton-Santilli over the conventional Hamiltonian mechanics. In fact, quantum mechanics can only represent (in first quantization) point-like particles under action-at-a-distance interactions. By comparison, hadronic mechanics can represent (in first isoquantization) the actual nonspherical shape of hadrons, their

deformations as well as their nonlocal-integral interactions due to mutual penetrations of the hadrons. The possibilities for broader applications in various disciplines are then evident.

II.2.14: Isolinearity, isolocality, isocanonicity and isounitarity

In Sect. II.1 we pointed out that the primary limitations of the contemporary formulation of Lie's theory are those of being linear, local and canonical. The classical and operator realizations identified earlier indicate rather clearly that the Lie-Santilli isotheory is nonlinear, nonlocal and noncanonical, as desired.

It is important to understand that such nonlinearity, nonlocality and noncanonicity occur only when the theory is projected in the original space over the original fields because the theory reconstructs linearity, locality and canonicity in isospaces over isofields (see [115] for all details and references).

Let $S(x,F)$ be a conventional vector space with local coordinates x over a field F, and let $x' = A(w)x$ be a linear, local and canonical transformation on $S(x,F)$, $w \in F$. The lifting $S(x,F) \rightarrow \hat{S}(\hat{x},\hat{F})$ requires a corresponding necessary isotopy of the transformations [52]

$$\hat{x}' = \hat{A}(\hat{w}) \hat{\times} \hat{x} = \hat{A}(\hat{w}) \times \hat{T} \times \hat{x}, \quad \hat{T} \text{ fixed}, \quad \hat{x} \in \hat{S}(\hat{x},\hat{F}), \qquad \hat{w} = w\hat{1} \in \hat{R}, \ \hat{1} = \hat{T}^{-1}, \tag{2.169}$$

called *isotransforms*, with *isodual isotransforms* $\hat{x}' = \hat{A}^d(\hat{w}^d \hat{\times}^d \hat{x} = - \hat{A}(\hat{w}) \hat{\times} \hat{x}$.

It is easy to see that the above isotransforms satisfy the condition of linearity in isospaces, called *isolinearity*

$$\hat{A} \hat{\times} (\hat{a} \hat{\times} \hat{x} + \hat{b} \hat{\times} \hat{y}) = \hat{a} \hat{\times} (\hat{A} \hat{\times} \hat{x}) + \hat{b} * (\hat{A} \hat{\times} \hat{y}), \ \forall \hat{x}, \hat{y} \in \hat{S}(\hat{x},\hat{F}), \hat{a}, \hat{b} \in \hat{F}, \tag{2.170}$$

although their projection in the original space $S(x,F)$ are nonlinear because $x' = \hat{A}T(x, \dot{x}, \ldots)x$.

Lemma 2.3 [116]: *All possible (well behaved) nonlinear, classical or operator systems of equations or of transformations always admit an identical isolinear reformulation*

The above property illustrates the primary mechanisms according to which the Lie-Santilli isotheory applies to nonlinear systems. In fact, as we shall see shortly, the latter theory is isolinear and, as such, it is capable of turning conventionally nonlinear systems into identical forms which do verify the axioms

of linearity in isospace, with evident advantages.

Isotransforms (2.74) are also *isolocal* in the sense that the theory formally deals with the local variables x while all nonlocal terms are embedded in the isounit, namely, all nonlocal-integral terms disappear at the abstract, realization-free level. Nevertheless, the theory is nonlocal when projected in the original space.

Similarly, isotopic theories are *isocanonical* because they are derivable from the isoaction (2.124) which is of canonical first-order type in isospace and coincides at the abstract level with the canonical action.

Finally, nonunitary transforms on $\mathcal{H}$, $U \times U^\dagger \neq I$, can always be identically rewritten as the following *isounitary transformations* [101]

$$U = \hat{U} \times \hat{T}^{1/2}, \quad U \times U^\dagger = \hat{U} \hat{\times} \hat{U}^\dagger = \hat{U}^\dagger \hat{\times} U = \hat{1}, \tag{2.171}$$

As a matter of fact, any conventionally nonunitary operator U, $U \times U^\dagger = \hat{1} \neq I$, on $\mathcal{H}$ always admits an identical isounitary form on $\hat{\mathcal{H}}$ via the simple rule $U = \hat{U}\hat{T}^{1/2}$.

II.3. LIE-SANTILLI ISOTHEORY AND ITS ISODUAL

II.3.1. Statement of the problem.

As recalled in Sect. II.1, Lie's theory (see, e.g., [15] for a mathematical presentation and [13] for a physical formulation) is centrally dependent on the basic n-dimensional unit I = diag. (1, 1, ..., 1) in all its major branches, such as enveloping algebras, Lie algebras, Lie groups, representation theory, etc.

The main idea of the *Lie-Santilli isotheory* [52], [53], [110] is the reformulation of the entire conventional theory with respect to the most general possible, integro-differential isounit $\hat{1}(x, \dot{x}, \ddot{x}, ...) = \hat{1}^\dagger$ (Fig. 3.1). The *Lie-Santilli genotheory* [loc. cit.] occurs when the Hermiticity of the theory is relaxed, $\hat{1} \neq \hat{1}^\dagger$, and the *hypertheory* [105], [106] occurs when, in addition to relaxing the Hermiticity, the generalized unit is multivalued. This paper is primarily devoted to the isotheory with only marginal comments on the broader genotheory and hypertheory.

The following introductory comments are in order. We should note from the outset the richness and novelty of the isotopic theory. In fact, the conventional Lie theory has only one formulation. By comparison, the Lie-Santilli isotheory can be

classified into five main classes I, II, III, IV and V as occurring for isofields, isospaces, etc., and admits novel realizations and representations.

THE STRUCTURE OF LIE'S THEORY

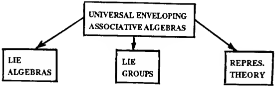


FIGURE 3.1: A reproduction of Fig. 4.1.1, p. 124, ref. [115] illustrating that the isotopies of Lie's theory are *not* studied within the context of *nonassociative algebras*, but are based instead on the *isotopies of the enveloping associative algebra*, from which the entire isotheory can be constructed, including isoalgebras, isogroups, isorepresentations, etc., as conceived in Santilli's original proposal [52]. The dominant motivation of the proposal was of purely *physical* character and consisted in achieving methods for the construction of the most general possible *nonlinear, nonlocal, and noncanonical transformations groups and symmetries* in such a way to preserve the abstract axioms of the contemporary *linear, local, and canonical Lie transformation groups and symmetries.* This unity of mathematical thought then permitted Santilli to preserve the abstract axioms of conventional physical laws also for structurally broader systems. In particular, the isotopies of Lie's theory permit the applicability of the abstract axioms of Galilei's and Einstein's relativities also for nonlinear, nonlocal and noncanonical systems. Mathematicians should be aware that the use of other generalizations of Lie's theory (e.g., quantum deformations) imply the loss of the above unity. In turn, this implies the violation of conventional relativities by the generalized formulations, thus creating the rather sizable problems of first identifying their replacements and then establishing them experimentally.

Second, we should point out the *inequivalence* of the conventional and isotopic formulation of Lie's theory in the following sense. Recall that, when formulated via Hermitean operators on a Hilbert space, Lie's theory is inclusive of all equivalence classes characterized by unitary transforms which, evidently preserve the fundamental unit, $U \times U^\dagger = U^\dagger \times U = I$. Being based on a generalized unit $\hat{1} \neq I$, *the Lie-santilli isotheory is therefore outside the equivalence classes of the*

conventional formulation. In fact, the former is derivable via *nonunitary transforms* of the latter,

$$
\begin{aligned}
&U \times U^{T} = \hat{1} = \hat{1}^{\dagger} \neq I, \quad \hat{T} = (U \times U^{\dagger})^{-1} = \hat{T}^{\dagger}, \\
&U \times I \times U^{\dagger} = \hat{1}, \qquad U \times A \times B \times U^{\dagger} = A' \times \hat{T} \times B \\
&U \times (A \times B - B \times A) \times U^{\dagger} = A' \times \hat{T} \times B' - B' \times \hat{T} \times A' \\
&A' = U \times A \times U^{\dagger}, \; B' = U \times B \times U^{\dagger},
\end{aligned} \tag{3.1}
$$

where one should note that the isounit and isotopic element have the correct Hermiticity property and interconnection.

The above inequivalence has rather profound implications, such as the fact that *weights and, more generally, the representation theories of the Lie and Lie-Santilli theories are inequivalent.* This can be see from the fact that nonunitary transforms do not preserve weight, e.g., because they do not preserve eigenvalues

$$
\begin{aligned}
&H \times |\psi> = E \times |\psi> \;\rightarrow\; H' \times \hat{T} \times |\hat{\psi}> = E' \hat{\times} |\hat{\psi}>, \quad E' \neq E, \\
&H' = U \times H \times U^{\dagger} = H'^{\dagger}, \; E' = U \times E \times U^{\dagger}, \; |\hat{\psi}> = U \times |\hat{\psi}>.
\end{aligned} \tag{3.2}
$$

The nontriviality of the Lie and Lie–Santilli theories is then illustrated by the fact that familiar spectra of eigenvalues, such as the *discrete spectrum* of the rotational group SO(3) $E = 0, 1, 2, \ldots$ is lifted into a *generally continuous spectrum* $E' = U \times E \times U^{\dagger}$ *of* the isorotational group $\hat{SO}(3)$, as we shall illustrate in section 3.6, even though the two groups are locally isomorphic for Class I isotopies, $SO(3) \approx \hat{SO}(3)$.

The apparent discrepancy caused by *isomorphic* groups with *inequivalent* representations is instructive for the reader first exposed to isotopies. In fact, conventional theories admit only *one* formulation, while all isotopic theories admit *two* different formulations, the first on isospaces over isofields, and the second given by their *projection* on conventional spaces over conventional fields.

As shown by Santilli in ref.s [115], [116], *when the isotopic groups are formulated on their appropriate isospaces over isofields, they coincide with the conventional groups, including the identity of the preservations and related weights, while inequivalent representations and weights emerge only when the isogroups are projected in carrier spaces and fields of the conventional groups.*

The above occurrence becomes clear if one recalls that *the map from groups to isogroups is characterized by the underlying map of the unit*, $I \rightarrow \hat{1}$.

Therefore, conventional weights E and their isotopic images $E' = U \times E \times U^\dagger$ are manifestly inequivalent when *both* are considered with respect to the *same* unit I. However, the weights $E = E \times I$ and $E' = E \times \hat{1} = E \times (U \times U^\dagger)$ are manifestly equivalent when *each* is represented with respect to *its own unit* I and $\hat{1} = U \times U^\dagger$, respectively.

Despite the above *mathematical* equivalence, isorepresentation have far reaching *physical* implications. This is due to the fact that physical events occur in our space-time and not in isospace. As a result, the physically significant isorepresentations are the *projections* in the original Lie space, and their formulation in isospace has purely mathematical significance.

To avoid ambiguities and misrepresentations, the correct formulation of the Lie-Santilli isotheory therefore requires the identification of the underlying spaces and fields. Throughout this section, unless otherwise stated, the Lie theory is formulated in the conventional spaces and expressed via conventional symbols, I, A, B, ..., A×B, etc., while the isotheory is formulated in isospaces over isofields and expressed with the symbols $\hat{1}$, $\hat{A}$, $\hat{B}$, ..., $\hat{A} \hat{\times} \hat{B}$, etc. When using *conventional* symbols for the *isotopic* theory we means its projection in conventional spaces.

The next topic which warrants advance comments is the *invariance* of the Lie-Santilli isotheory within its own isotopic context. We have recalled earlier that the conventional unit I is the fundamental quantity of the conventional Lie's theory verifying the familiar properties $I^n = I \times I \times \ldots \times I \equiv I$, $I^{\frac{1}{2}} = I$, $I/I = I$, etc. Moreover, the unit is a *trivial first integral* of the equations of motion, e.g., $i\, dI/dt = [I, H] = I \times H - H \times I = H - H \equiv 0$. Moreover, the conventional unit is *invariant* under the group transformations it characterizes, e.g., for unitary transforms we have $U \times I \times U^\dagger \equiv U^\dagger \times I \times U \equiv I$.

In Sect. 2 we have shown that the fundamental unit of the Lie-Santilli isotheory, the isounit $\hat{1}$, does indeed preserve the axiomatic properties of the conventional unit, and it does indeed remain a first integral of the equations of motion,

$$\hat{1}^{\hat{n}} = \hat{1} \hat{\times} \hat{1} \hat{\times} \ldots \hat{\times} \hat{1} \equiv \hat{1}, \quad \hat{1}^{\hat{\frac{1}{2}}} \equiv \hat{1}, \quad \hat{1} \hat{/} \hat{1} \equiv \hat{1}, \quad \text{etc.}$$

$$i\, \hat{\partial} \hat{1} / \hat{\partial} \hat{t} = \hat{1} \hat{\times} \hat{H} - \hat{H} \hat{\times} \hat{1} = \hat{H} - \hat{H} \equiv 0 \,. \tag{3.3}$$

What remains to point out is its invariance. In this respect, it is easy to see that *the isounit is not invariant both unitary as well as* nonu*nitary transforms*. In fact, under a unitary transform, $U \times U^\dagger = U^\dagger \times U = I$, we have $U \times \hat{1} \times U^\dagger = \hat{1}' \neq \hat{1}$ and, similarly, under a nonunitary transform, $W \times W^\dagger \neq I$, we have $W \times \hat{1} \times W^\dagger = \hat{1}' \neq \hat{1}$.

However, *the isounit is invariant under isounitary transforms* (Sect. 2.6). In fact, any nonunitary transform $W \times W^\dagger \neq I$ can always be identically written in the isounitary form

$$W = \hat{W} \times \hat{T}^{1/2}, \qquad W \times W^{\dagger} = \hat{W} \hat{\times} \hat{W}^{\dagger} = W^{\dagger} \times W = \hat{W}^{\dagger} \hat{\times} \hat{W} = \hat{1}, \tag{3.4}$$

under which we have the invariance, not only of the isounit, but also of the isotopic products

$$\hat{W} \times \hat{1} \hat{\times} \hat{W}^{\dagger} \equiv \hat{1}, \quad \hat{W} \hat{\times} \hat{A} \times \hat{B} \hat{\times} \hat{W}^{\dagger} = \hat{A}' \hat{\times} \hat{B}',$$

$$\hat{W} \hat{\times} (\hat{A} \hat{\times} \hat{B} - \hat{B} \hat{\times} \hat{A}) \hat{\times} \hat{W}^{\dagger} = \hat{A}' \hat{\times} \hat{B}' - \hat{B}' \hat{\times} \hat{A}',$$

$$\hat{A}' = \hat{W} \hat{\times} \hat{A} \hat{\times} \hat{W}^{\dagger}, \quad \hat{B}' = \hat{W} \hat{\times} \hat{B} \hat{\times} \hat{W}^{\dagger}. \tag{3.5}$$

Note that the isounit and isotopic element are not only invariant, but left *numerically unchanged.*

The above occurrence illustrates once more the necessity of lifting the entire mathematical structure of Lie's theory for the correct formulation of the Lie–Santilli isotheory, without any exception known to this author.

It is remarkable that all the preceding properties persist under the broadening of the isotheory to its genotopic and hyperstructural coverings as the reader may verify.

II.3.2. Isoenvelopes and their isoduals.

In this section we study the *universal enveloping isoassociative algebras* (or *isoenvelopes* for short) for the case of Class III over an isofield of characteristic zero of the same class, as first formulated by Santilli in memoir [52] of 1978 and then presented in monograph [110] (for independent studies see [25], [121]). The use of Class III implies a unified formulation of the isotopies of Classes I and II and permits the unification of the envelopes of simple, compact and noncompact Lie algebras of the same dimension into one single isotope.

To begin, let $\xi = \xi(L)$ be a universal enveloping associative algebra of an N-dimensional Lie algebra **L** (see, e.g., ref. [15]) with generic elements A, B, C,... , trivial associative product $A{\times}B = AB$ (say, of matrices) and unit matrix in N-dimension $I = \mathrm{diag.}(1, 1, ..., 1)$.

Let the (ordered) basis of **L** be given by $\{X_k\}$, $k = 1, 2, ..., N$, over a field $F(a,+,\times)$. An (ordered) *standard monomial* of dimension n is the product of n-generators $X_i{\times}X_j{\times} ... {\times}X_k$ with the ordering $i \leq j \leq ... \leq k$. The infinite-dimensional basis of $\xi(L)$ is then expressible in terms of monomial and given by the *Poincaré–Birkhoff–Witt theorem* [loc. cit.]

$$I, \quad X_k \quad X_i \times X_j \quad (i \leq j), \qquad X_i \times X_j \times X_k \quad (i \leq j \leq k), \quad ...... \tag{3.6}$$

The *universal enveloping isoassociative algebra*, or *isoenvelope*) $\hat{\xi}(L)$ of the *Lie* algebra L [52] (see Fig. 3.2) coincide with ξ as vector spaces (because the basis of a vector space is unchanged under isotopies). The basis of $\hat{\xi}(L)$ is therefore constructed with the same generators X_k only computed on the new isospace $\hat{S}(\hat{x},\hat{F}(\hat{a},+,\hat{\times}))$, denoted $\hat{X}_k$ and now equipped with the isoproduct $\hat{A}\hat{\times}\hat{B}$ so as to admit $\hat{1} = \hat{T}^{-1}$ as the correct (right and left) unit

$$\hat{\xi} \;:\; \hat{A}\hat{\times}\hat{B} = \hat{A}\times\hat{T}\times\hat{B} = \hat{A}\hat{T}\hat{B}, \quad \hat{T} \text{ fixed},$$

$$\hat{1}\hat{\times}\hat{A} = \hat{A}\hat{\times}\hat{1} \equiv \hat{A} \qquad \forall \hat{A} \in \hat{\xi}, \quad \hat{1} = \hat{T}^{-1}. \tag{3.7}$$

The (ordered) standard monomials of dimension N of $\xi(L)$ are then mapped into the (ordered) *standard isomonomials* of the same dimension $\hat{X}_i\hat{\times}\hat{X}_j\hat{\times} ... \hat{\times}\hat{X}_k$, $i \leq j \leq ... \leq k$ of $\hat{\xi}(L)$.

A fundamental property from which most of the Lie-isotopic theory can be derived is the following

Theorem 3.1 (Poincaré-Birkhoff-Witt-Santilli Theorem [52], [110]): *The cosets of* $\hat{1}$ *and the standard isomonomials form an infinite-dimensional basis of the universal enveloping isoassociative algebra* $\hat{\xi}(L)$ *of a Lie algebra* L *of Class III*

$$\hat{1}, \quad \hat{X}_k \quad \hat{X}_i\hat{\times}\hat{X}_j \quad (i \leq j), \qquad \hat{X}_i\hat{\times}\hat{X}_j\hat{\times}\hat{X}_k \quad (i \leq j \leq k), \quad ...... \tag{3.8}$$

A detailed proof can be found in ref. [110], pp. 154-163, or ref. [121], pp. 74-93, and it is not repeated here for brevity (although its knowledge is assumed for more advanced treatments).

Algebraically, the above theorem essentially expresses the property that non singular isotopies of the basic product, i.e.,

$$A \times B: \quad (A{\times}B){\times}C = A{\times}(B{\times}C) \rightarrow A\hat{\times}B: \quad (A\hat{\times}B)\hat{\times}C = A\hat{\times}(B\hat{\times}C), \tag{3.9}$$

imply the existence of consistent isotopies of the basis. Note the abstract unity of the conventional and isoenvelopes. In fact, at the level of realization-free formulation the "hat" can be ignored and bases (3.6) and (3.8) coincide. Nevertheless, the isoenvelope $\hat{\xi}(L)$ is structurally broader then the conventional envelope $\xi(L)$, e.g., because it unifies compact and noncompact structures as shown below, and this

begins to illustrate the nontriviality of the Lie–Santilli isotheory.

Theorem 3.1 and isobasis (3.8) have important mathematical and physical implications. Recall that the conventional exponentiation is defined via a power series expansions in ξ

$$e^{iwX} = e_\xi^{iwX} = 1 + (i\,w{\times}X)/1! + (i\,w{\times}X) \times (i\,w{\times}X)/2! + \ldots\ldots, \quad w \in F(a,+,\times). \tag{3.10}$$

The above exponentiation is then inapplicable under isotopies because the quantity 1 is no longer the basic unit of the theory, the conventional product $\times$ has no mathematical or physical meaning, etc.

In turn, this implies that all quantum mechanical quantities depending on the conventional exponentiation, such as time evolution, unitary groups, Dirac's delta distributions, Fourier transforms, Gaussian, etc. have no mathematical or physical meaning under isotopies and must be suitably lifted.

Isobasis (3.8) then permits the following

Corollary 3.1.A [52]: *The "isoexponentiation" of an element* $X \in \hat{\xi}$ *via isobasis* (3.8) *over an isofield* $\hat{F}(\hat{a},+,\hat{\times})$ *is given by*

$$\hat{e}^{i\,\hat{w}\,\hat{\times}\,\hat{X}} \equiv \hat{e}^{i\,w\,X} = e_{\hat{\xi}}^{i\,\hat{w}\,\hat{\times}\,\hat{X}} \equiv e_{\hat{\xi}}^{i\,w\,X} =$$

$$= \hat{1} + (i\,\hat{w}\,\hat{\times}\,\hat{X})/1! + (i\,\hat{w}\,\hat{\times}\,\hat{X})\,\hat{\times}\,(i\,\hat{w}\,\hat{\times}\,\hat{X})/2! + \ldots\ldots =$$

$$= \hat{1} \times \{e^{iw\times T\times X}\} = \{e^{iX\times T\times w}\} \times \hat{1}, \qquad \hat{w} \in \hat{F}(\hat{a},+,\hat{\times}).$$

$$\hat{w}\,\hat{\times}\,\hat{X} = (w \times \hat{1}) \times \hat{T} \times \hat{X} \equiv w\,X, \quad \hat{X}_{S(\hat{x},\hat{F}(\hat{a},+,\hat{\times}))} \equiv X_{S(x,F(a,+,\times))}. \tag{3.10}$$

The nontriviality of the isotopies of Lie's theory is clearly expressed by *the appearance of the nonlinear, nonlocal and noncanonical isotopic element* $\hat{T}(t, x, \dot{x}, \ddot{x}, \ldots)$ *directly in the exponent of isoexponentiations* (3.10). This is sufficient to see that the Lie–isotopic space-time and internal symmetries are nonlinear, nonlocal and noncanonical, as desired.

One should keep in mind the *uniqueness of isoexponentiation* (3.10) which implies the uniqueness of related physical laws This property should be compared with the *lack of uniqueness* of the exponentiation in other theories, e.g. the so-called q-deformations.

The *isodual isoenvelopes* $\hat{\xi}^d(\hat{L}^d)$ [62], [115] are characterized by: the *isodual basis* and the *isodual parameters*

$$\hat{X}^d{}_k = -\hat{X}_k, \quad \hat{w}^d = w\,\hat{1}^d = -\hat{w}. \tag{3.11}$$

Corollary 3.1.B: *The "isodual isoexponentiation" is the isodual image of isoexponentiation (4.3.6) on the isodual isofield* $\hat{F}^d(\hat{w}^d,+,\hat{\times}^d)$

$$\hat{e}^{d\ \mathrm{i}^d \hat{w}^d \hat{\times}^d \hat{X}^d} = e_{\hat{\xi}^d}^{\ \mathrm{i}^d \hat{w}^d \hat{\times}^d \hat{X}^d} = -\{ e^{\mathrm{i} X T w} \} \hat{1} \tag{3.12}$$

Note that the preservation of the sign in the exponent is only apparent, i.e., when projected in an isofield, because, when properly written in the isodual isofield, one can use the expression

$$e_{\hat{\xi}^d}^{\ -\mathrm{i}^d \hat{w}^d \hat{\times}^d \hat{X}^d} \equiv -\{ e^{-\mathrm{i} X T w^d} \} \hat{1} \tag{3.13}$$

Isodual isoexponentiations play an important role for the construction of the isodual isosymmetries for antiparticles. The following property is trivial from the analysis of Sect. 3.1

Corollary 3.1.C: *Envelopes are isoenvelopes are not unitarily equivalent.*

It is easy to see that Theorem 3.1 holds for envelopes of Class III, as originally formulated [52], thus unifying isoenvelopes $\hat{\xi}$ and their isoduals $\hat{\xi}^d$ and permitting the unified representation of nonisomorphic Lie algebras of the same dimension. To clarify this aspect, recall that a conventional envelope $\xi(L)$ represents only one algebra (up to local isomorphism),

$$L \approx [\xi(L)]^- . \tag{3.14}$$

The study of a nonisomorphic Lie algebras then requires the use of a *different basis* X'_k, resulting in a *different envelope* $\xi'(L')$. Thus, in the conventional Lie theory *nonisomorphic Lie algebras of the same dimension are represented via different bases and different envelopes.*

This scenario is altered under isotopy because the isoenvelopes are now characterized by *two* quantities, the basis $\hat{X}_k$ *and* the isounit $\hat{1}$. We therefore have the novel possibility of using the *same* basis and changing instead the *isounit.* In fact, one isoenvelope $\hat{\xi}(L)$ of Class III with a fixed N-dimensional basis X_k and an arbitrary N-dimensional isounit $\hat{1}$ represents a family of generally nonisomorphic Lie algebras $\hat{L}$ as the attached antisymmetric algebras

$$\hat{L} \approx [\hat{\xi}(L)]^- . \tag{3.15}$$

In particular, it was proved in the original proposal [52] that, the isoalgebra $\hat{L}$ constructed via the above rule *is not* in general isomorphic to the original algebra L, $\hat{L} \not\approx L$, unless the isotopic element is positive-definite.

Theorem 3.1 therefore offers the possibility of unifying of all simple Lie algebra in Cartan's classification of the same dimension which was presented as a *conjecture*. This would imply in particular the reduction of compact and noncompact structures of the same dimension to only one isotopic structure, and, for each given structure, the reduction of linear and nonlinear, local and nonlocal, canonical and noncanonical realizations to one primitive algebraic notion, the isoenvelope $\hat{\xi}(L)$ (see Fig. 3.2 below for more details).

The above conjecture was illustrated in the original proposal [52] with an example that is still valid today. Consider the conventional Lie algebra **so(3)** of the rotational group SO(3) on the Euclidean space E(r,δ,R) with unit I = diag. (1, 1, 1). The adjoint representation of **so(3)** is given by the familiar expressions

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.16}$$

The universal enveloping associative algebra ξ(**so(3)**) is then characterized by the unique infinite-dimensional basis from the conventional Poincaré-Birkhoff-Witt theorem [15]

$$I, \quad J_k, \quad J_i \times J_j \ (i \leq j), \quad J_i \times J_j \times J_k \quad (j_i \leq j \leq k), \ldots \tag{3.17}$$

and characterizes only one algebra as the attached antisymmetric algebra

$$[\, \xi\, (\,\mathbf{so(3)}\,)\,]^{-} \approx \mathbf{so(3)}\,. \tag{3.18}$$

The isotopies $\hat{\xi}$(**so(3)**) of the envelope ξ(so(3)) of Class III are characterized by the the lifting of the basic carrier space E(r,δ,R) into the isoeuclidean space $\hat{E}(\hat{r},\hat{\delta},\hat{R})$ with isometric, isotopic element and isounit

$$\hat{\delta} = \hat{T}\delta, \ \hat{T} = \text{diag.}\,(g_{11}, g_{22}, g_{33}), \ \hat{1} = \text{diag.}\,(g_{11}{}^{-1}, g_{22}{}^{-1}, g_{33}{}^{-1}), \tag{3.19}$$

where the characteristic quantities g_{kk} are real-valued, non-null but otherwise arbitrary functions of the local coordinates $g_{kk}(t, r, \dot{r}, \ddot{r}, \ldots)$ which, as such, can be either positive or negative. From Theorem 3.1, the isoenvelope $\hat{\xi}$(**so(3)**) is then characterized by the original generators (3.16) although expressed now in terms of the isoassociative product $J_i \hat{\times} J_j = J_i \times \hat{T} \times J_j$ and isounit $\hat{1}$ with unique infinite-

dimensional basis

$$\hat{1}, \quad J_k, \quad J_i \times T \times J_j \ (i \leq j), \quad J_i \times T \times J_j \times T \times J_k \quad (j_i \leq j \leq k), \ldots \tag{3.20}$$

It is now easy to see that the algebra characterized by the attached antisymmetric part of $\xi(so(3))$ *is not unique*, evidently because it depends on the explicit values of the characteristic quantities g_{kk}. It was shown in ref.s [52, [110] that the isoenvelope $\xi(so(3))$ unifies: all possible compact and noncompact three-dimensional Lie algebra of Cartan classification, the algebras **so(3)** and **so(2.1)**; all their infinitely possible isotopes $\hat{so}(3)$ and $\hat{so}(2.1)$; the compact and noncompact isodual algebras $so^d(3)$ and $so^d(3)$; as well as all their infinitely possible isodual isotopes $\hat{so}^d(3)$ and $\hat{so}^d(2.1)$, according to the classification

$$[\xi(so(3))]^{\Gamma}: \quad \begin{array}{l} so(3) \text{ for } T = \text{diag.}\,(1, 1, 1); \\ so(2.1) \text{ for } T = \text{diag.}\,(1, -1, 1); \\ \hat{so}(3) \text{ for sign. } T = (+, +, +); \\ \hat{so}(2.1) \text{ for sign. } T = (+, -, +) \\ so^d(3) \text{ for } T = (-1, -1, -1); \\ so^d(2.1) \text{ for } T = \text{diag.}\,(-1, +1, -1); \\ \hat{so}^d(3) \text{ for sign. } T = (-, -, -); \\ \hat{so}^d(2.1) \text{ for sign. } T = (-, +, -). \end{array} \tag{3.21}$$

The unification of all simple Lie algebras of dimension 6 in Cartan's classification was also identified by Santilli in ref. [59] and it will be studied later on. The general case of isotopic unification is studied by Tsagas [124].

The isoenvelopes are denoted $\xi(L)$ and *not* $\xi(\hat{L})$ to stress the preservation of the original basis of L under isotopies, as well as to emphasize the existence of an infinite family of isoenvelopes for *each* original Lie algebra L.

The isoenvelope outlines above was further developed by Santilli [115] into the *genoenvelope* in which the product is ordered either to the right or to the left, resulting in two different structures

$$\xi^>: \ \hat{1}^> = \hat{S}^{-1}, \ X_i > X_j, i \leq j, \ X_i > X_j > X_k, i \leq j \leq k, \ X_i > X_j = X_i \times \hat{S} \times X_j,$$

$$^<\xi: \ ^<\hat{1} = \hat{R}^{-1}, \ X_i < X_j, i \leq j, \ X_i < X_j < X_k, i \leq j \leq k, \ X_i < X_j = X_i \times \hat{R} \times X_j, \tag{3.22}$$

interconnected by the conjugation $\xi^> = (\ ^<\xi\ )^\dagger$, and defined over the respective

fields to the right and to the left (Sect. 2.3). The important property is that, as the reader can verify, each of the above two envelopes verifies the Poincaré–Birkhoff–Witt–Santilli Theorem.

In particular, the above infinite bases permit the definition of *two* exponentiations, one to the right and one to the left

$$\hat{e}^{>\, i\, X\times w} = \hat{1}^{>} + (iX\times w)/1! + (iX\times w) > (iX\times w)/2! + \ldots\ldots = (e^{iX\times\hat{S}\times w})\times\hat{1}^{>},$$

$$^{<}\hat{e}^{\, i\, w\times X} = {}^{<}\hat{1} + (i\, w\times X)/1! + (i\, w\times X) < (i\, w\times X)/2! + \ldots\ldots = {}^{<}\hat{1}\times(e^{iw\times\hat{R}\times X}), \tag{3.23}$$

with similar dual genotopies of the remaining aspects.

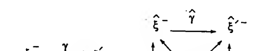
The genoenvelopes, in turn, can be further enlarged into the *hyperenvelopes* [106] in which the generalized units and multiplications are not only ordered to the right and to the left but are also multivalued,

$$\{\hat{\xi}^{>}\}: \ \{\hat{1}^{>}\} = \{\hat{S}\}^{-1}, \ \{X_i\}\{>\}\{X_j\} = \{X_i\}\times\{\hat{S}\}\times\{X_j\}, \ i \leqq j, \quad \text{etc.}$$

$$\{^{<}\hat{\xi}\}: \ \{^{<}\hat{1}\} = \{\hat{R}\}^{-1}, \ \{X_i\}\{<\}\{X_j\} = \{X_i\}\times\{\hat{R}\}\times\{X_j\}, \ i \leqq j, \quad \text{etc.} \tag{3.24}$$

where { ... } denotes a finite and ordered set as in Sect.s 2.1 and 2.2. [115].

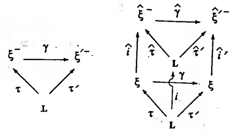
UNIVERSAL ISOASSOCIATIVE ENVELOPING ALGEBRAS



(c) (d)

FIGURE 3.2: A reproduction of Fig. 4.3.1, p. 140, ref. [115]. The *universal enveloping associative algebra* ξ(L) of a Lie algebra L [15] is the set (ξ, τ) where ξ is an associative

algebra and τ is a homomorphism of **L** into the antisymmetric algebra ξ^- attached to ξ such that: if ξ' is another associative algebra and τ' is another homomorphism of **L** into ξ'^- a unique isomorphism γ between ξ and ξ' exists in such a way that the diagram (a) above is commutative. The above definition evidently expresses the *uniqueness* of the Lie algebra L (up to local isomorphisms) characterized by its universal envelopes $\xi(L)$. With reference to diagram (b) above, the *universal enveloping isoassociative algebra* $\hat{\xi}(L)$ of a Lie algebra **L** was introduced [52] as the set $\{(\xi, \tau), \imath, \hat{\xi}, \hat{\tau}\}$ where: (ξ, τ) is a conventional envelope of L; $\imath$ is an isotopic mapping $\mathbf{L} \rightarrow \imath\,\mathbf{L} = \hat{\mathbf{L}} \not\approx \mathbf{L}$; $\hat{\xi}$ is an associative algebra generally nonisomorphic to ξ; $\hat{\tau}$ is a homomorphism of $\hat{\mathbf{L}}$ into $\hat{\xi}^-$; such that: if $\hat{\xi}'$ is another associative algebra and $\hat{\tau}'$ another homomorphism of $\hat{\mathbf{L}}$ into $\hat{\xi}'^-$, there exists a unique isomorphism $\hat{\gamma}$ of $\hat{\xi}$ into $\hat{\xi}'$ with $\hat{\tau}' = \hat{\gamma}\hat{\tau}$, and two unique isotopies $\hat{\imath}\xi = \hat{\xi}$ and $\hat{\imath}'\xi' = \hat{\xi}'$. A primary objective of the isotopic definition is the achievement of *the lack of uniqueness of the Lie algebra characterized by the isoenvelope* or, equivalently, the characterization of a family of generally nonisomorphic Lie algebras via the use of only one basis. The illustration of the above notions for the case of the rotational algebra so(3) studied in the text is straighforward and can be expressed via the diagrams (c) and (d) below



(a) (b)

where the isotopy is given by $1 = \mathrm{diag.}\,(1, 1, 1) \rightarrow \hat{1} = \mathrm{diag.}\,(1, -1, 1)$. The above definition then provides all infinitely possible isotopes and isodual isotopes.

We can the introduce the *hyperexponentiations to the right and to the left*

$$\{\hat{e}^{>}\}^{\imath\, w \times \{X\}} = (\{e\}^{\imath \{X\} \times \{\hat{S}\} \times w}) \times \hat{1}^{>},$$

$$\{{}^{<}\hat{e}\}^{\imath\, w\, X} = ({}^{<}\hat{1} \times (\{e\}^{\imath w \times \{\hat{R}\} \times \{X\}}). \tag{3.25}$$

The interested reader can then work out the remaining aspects of the hyperenvelopes.

Note that *envelopes* to the right and to the left are trivially equivalent and the same happens for the *isoenvelopes*. On the contrary, *genoenvelopes* to the right and to the left are not equivalent and the same happens for the *hyperenvelopes*. This illustrates the emergence in the latter cases of a much broader representation theory. In fact, conventional envelopes can be studied via *one-sided moduli* (yielding the ordinary *representations*), the isoenvelopes require *one-sided moduli* (yielding the *isorepresentations*), while genoenvelopes require *two-sided bimoduli* (yielding the *birepresentations*), which are not reducible to a one-sided form as in the preceding cases, and the hyperenvelopes require *two-sided hypermoduli* (yielding the *multivalued byrepresentations*).

Recall the conjecture that all possible Lie algebras of Cartan's classification *with the same dimension* may be characterized by one single isoenvelope (or genoenvelope) [52], [124]. We then have the *conjecture* that *hyperenvelopes may unify all possible simple Lie algebra of the Cartan classification of arbitrary dimension into one single structure* [115].

For additional studies we refer the reader to monographs [115], [121].

II.3.3. Lie-Santilli isoalgebras and their isoduals.

We are now equipped to introduce the following

Definition 3.1 [52], [115]: *A (finite-dimensional) isospace $\hat{L}$ over an isofield $\hat{F}(\hat{a},+,\hat{\times})$ of isoreal numbers $\hat{R}(\hat{n},+,\hat{\times})$, isocomplex numbers $\hat{C}(\hat{c},+,\hat{\times})$ or isoquaternions $\hat{Q}(\hat{q},+,\hat{\times})$ with isotopic element $\hat{T}$ and isounit $\hat{1} = \hat{T}^{-1}$ is called a "Lie-Santilli isoalgebra" over $\hat{F}$ when there is a composition $[\hat{A},\hat{}\hat{B}]$ in $\hat{L}$, called "isocommutator", which verifies the following "isolinear and isodifferential rules" for all $\hat{a}, \hat{b} \in \hat{F}$ and $\hat{A}, \hat{B}, \hat{C} \in \hat{L}$*

$$[\hat{a} \hat{\times} \hat{A} + \hat{b} \hat{\times} \hat{B} \hat{,} \hat{C}] = \hat{a} \hat{\times} [\hat{A} \hat{,} \hat{C}] + \hat{b} \hat{\times} [\hat{B} \hat{,} \hat{C}],$$

$$[\hat{A} \hat{\times} \hat{B} \hat{,} \hat{C}] = \hat{A} \hat{\times} [\hat{B} \hat{,} \hat{C}] + [\hat{A} \hat{,} \hat{C}] \hat{\times} \hat{B}. \tag{3.26}$$

and the "Lie-Santilli isoaxioms",

$$[\hat{A}\,\hat{,}\,\hat{B}] = -[\hat{B}\,\hat{,}\,\hat{A}],$$

$$[\hat{A}\,\hat{,}[\hat{B}\,\hat{,}\,\hat{C}]] + [\hat{B}\,\hat{,}[\hat{C}\,\hat{,}\,\hat{A}]] + [\hat{C}\,\hat{,}[\hat{A}\,\hat{,}\,\hat{B}]] = 0. \tag{3.27}$$

Note that the use of isoreals, isocomplexes and isoquaternions preserves the associative character of the underlying envelope. The use instead of isooctonions $O(\hat{o},+,\hat{\times})$ (Sect. 2.3) would imply the loss of such an associative character and, for this reason, isooctonions have been excluded as possible isofields in Definition 3.1 in a way fully parallel to conventional lines in number theory. Nevertheless, one should keep in mind that the *Lõhmus-Paal-Sorgsepp octonionization process* [31] resolves the above problematic aspects.

In the original proposal [52] Santilli proved the existence of consistent *isotopic generalization of the celebrated Lie's First, Second and Third Theorems.* For brevity, we refer the interested reader to ref. [110], pp. 163–184 or to the ref. [121], Ch. II. We here quote the *Isotopic second and third Theorems* because useful for the speedy construction of realizations of isoalgebras (see later on for more other realizations).

Theorem 3.2 (Lie-Santilli Second Theorem [52], [110]): *Let* $X = \{X_k\}$, $k = 1, 2, \ldots, N$, *be the (ordered set of) generators in adjoint representations of a Lie algebra* **L** *with commutation rules*

$$\mathbf{L}:\ [X_i, X_j] = X_i \times X_j - X_j \times X_j = C_{ij}^{\ k}\, X_k\ , \tag{3.28}$$

where $C_{ij}^{\ k}$ *are the "structure constants". Then, one realization of the Lie-isotopic images* $\hat{\mathbf{L}}$ *of* **L** *is characterized by the same generators* $\hat{X}$ *now computed in isospaces over isofields with isocommutation rules*

$$\hat{\mathbf{L}}:\ [\hat{X}_i\,\hat{,}\,\hat{X}_j] = \hat{X}_i \hat{\times} \hat{X}_j - \hat{X}_j \hat{\times} \hat{X}_i = \hat{X}_i \times \hat{T} \times \hat{X}_j - \hat{X}_j \times \hat{T} \times \hat{X}_i =$$

$$= X_i \times T(x, \dot{x}, ..) \times X_j - X_j \times T(x, \dot{x}, ...) \times X_i = \hat{C}_{ij}^{\ k}(t, x, \dot{x}, \ldots) \hat{\times} \hat{X}_k =$$

$$= \hat{C}_{ij}^{\ k}(x, \dot{x}, \ldots) \times \hat{X}_k\,, \tag{3.29}$$

where the $\hat{C}_{ij}^{\ k}$ *are the "structure functions" in the isofield.*

Theorem 3.3 (Lie-Santilli Third Theorem [loc. cit.]): *The structure functions* $\hat{C}_{ij}^{\ k}$ *of an isoalgebra* $\hat{\mathbf{L}}$ *satisfy the conditions*

$$\hat{C}_{ij}^{\ k} = -\hat{C}_{ji}^{\ k}, \tag{3.30}$$

and the property (when commuting with the generators)

$$\hat{C}_{ij}{}^{p} \hat{\times} \hat{C}_{pk}{}^{q} + \hat{C}_{jk}{}^{p} \hat{\times} \hat{C}_{pi}{}^{q} + \hat{C}_{ki}{}^{p} \hat{\times} \hat{C}_{pj}{}^{q} = 0 . \tag{3.31}$$

It is important to illustrate the above theorems with an example. Consider the generators of the **su(2)** Lie algebra in their adjoint representation, which are given by the celebrated *Pauli's matrices* and related commutation rules

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$[\sigma_n , \sigma_m] = \sigma_n \times \sigma_m - \sigma_m \times \sigma_n = 2 i \epsilon_{nmk} \sigma_k , \tag{3.32}$$

Theorem 3.2 states that the *same generators* σ_k, when written in isospaces over isofields, can characterize *one realization* of the isoalgebra $\hat{\mathbf{su}}(2)$ via the lifting of the structure constants into suitable functions.

This property is readily verified by introducing a Class III isotopic element assumed diagonal for simplicity, and then identifying the structure functions under which the algebra is closed. By ignoring for notational simplicity the rewriting of the basis in isospace, we have the following illustration of the Lie–Santilli Second Theorem [115]

$$[\sigma_n \hat{,} \sigma_m] = \sigma_n \times T \times \sigma_m - \sigma_m \times T \times \sigma_n = 2 i \hat{\epsilon}_{nmk} \times T \times \sigma_k ,$$

$$T = \begin{pmatrix} g_{11} & 0 \\ 0 & g_{22} \end{pmatrix}, \; g_{kk} \neq 0, \;\; \Delta = \det T = g_{11}\, g_{22} ,$$

$$\hat{1} = \begin{pmatrix} g_{11}{}^{-1} & 0 \\ 0 & g_{22}{}^{-1} \end{pmatrix} = \Delta^{-1} \begin{pmatrix} g_{22} & 0 \\ 0 & g_{11} \end{pmatrix},$$

$$\hat{\epsilon}_{ijk} = \epsilon_{ijk} \begin{pmatrix} g_{22}/g_{11} & 0 \\ 0 & g_{11}/g_{22} \end{pmatrix} . \tag{3.33}$$

Note that the original structure "constants" $C_{ij}{}^{k}$ are elements of a field $F(a,+,\times)$ and, as such, are ordinary numbers. On the contrary, the structure

"functions" $\hat{C}_{ij}{}^{k}$ are now elements of the isofield $\hat{F}(\hat{a},+,\hat{\times})$ and, as such, are isonumbers and, thus, matrices. As such, they should be called more properly *structure isofunctions*, where the prefix "iso" stands precisely to represent their matrix character.

Note finally that Theorem 3.2 provide only one *method* for the speedy construction of an isotope $\hat{L}$ of a given Lie algebra **L** In general, the above methods is not applicable because Lie and Lie-isotopic algebras are connected by a nonunitary transform (Sect. 3.1), thus implying *different generators*. In fact, another way of constructing Class I isotopes $\hat{L}$ of a given Lie algebra **L** is by generalizing the generators X_k and keeping instead the old structure constants. This alternative approach is used in a number of applications because it ensures the local isomorphism $\hat{L} \approx$ **L** *ab initio*, while lifting original algebra into the desired nonlinear, nonlocal and noncanonical form.

It should be noted that Theorems 3.2 and 4.3.3 were conceived for specific physical needs. Recall that the generators of a Lie algebra represent physical quantities, such as linear momentum, angular momentum, energy, etc. As such, these quantities cannot be changed under isotopies, thus explaining the preservation of the original basis. An additional motivation is that, among all possible realizations, the method of Theorem 3.2 results to be most effective in the computation of the symmetries of nonlinear, nonlocal, noncanonical systems, as we shall see.

It is easy to prove the following:

Theorem 3.4 [110]: *The isotopies* L $\rightarrow \hat{L}$ *of an N-dimensional Lie algebra* **L** *preserve the original dimensionality.*

In fact, the basis e_k , k = 1, 2, ..., N of a vector space and, thus, of a Lie algebra **L** is not changed under isotopy, except for renormalization factors denoted $\hat{e}_k$. Let then the commutation rules of **L** be given by

$$[\, e_i \, , e_j \,] \;=\; C_{ij}{}^{k}\, e_k. \tag{3.34}$$

The isocommutation rules of the isotopes $\hat{L}$ are

$$[\hat{e}_i \hat{,} \hat{e}_j\,] = \hat{e}_i \times \hat{T} \times \hat{e}_j \;-\; \hat{e}_j \times \hat{T} \times \hat{e}_i \;=\; \hat{C}_{ij}{}^{k}(t, x, \dot{x}, \,...) \times \hat{e}_k\,. \tag{3.35}$$

One can see again in this way the necessity of lifting the structure "constants" into structure "functions", as correctly predicted by the Lie-isotopic Second Theorem.

We now review a few basic notions of Lie-isotopic algebras $\hat{L}$ which can be derived via an easy isotopy of the corresponding conventional notions, as first

studied by Kadeisvili [25]. The Lie-Santilli isoalgebras $\hat{L}$ are said to be:

a) *isoreal (isocomplex)* when $\hat{F} = \hat{R}$ ($\hat{F} = \hat{C}$);

2) *isoabelian* when $[\hat{A} \hat{,} \hat{B}] = 0$, $\forall$ A, B $\in \hat{L}$;

3) A subset $\hat{L}_0$ of $\hat{L}$ is said to be an *isosubalgebra* of $\hat{L}$ when

$$[\hat{L}_0 \hat{,} \hat{L}_0] \subset \hat{L}_0 ; \tag{3.36}$$

4) An *isoideal* occurs when

$$[\hat{L} \hat{,} \hat{L}_0] \subset \hat{L}_0; \tag{3.37}$$

5) The *isocenter* of a Lie-isotopic algebra is the maximal isoideal $\hat{L}_0$ which verifies the property

$$[\hat{L} \hat{,} \hat{L}_0] = 0. \tag{3.38}$$

Definition 3.2 [25]: *The "general isolinear and isocomplex Lie-Santilli algebras", denoted with* $\hat{GL}(n,\hat{C})$, *are the vector isospaces of all n×n complex matrices over* $\hat{C}(\hat{c},+,\hat{*})$, *and are evidently closed under isocommutators. The "isocenter" of* $\hat{GL}(n,\hat{C})$ *is then given by* $\hat{c}\hat{\times}\hat{1}$, $\forall\ \hat{c} \in \hat{C}$. *The subset of all complex n×n matrices with null trace is also closed under isocommutators, it is called the "special, isolinear, isocomplex, Lie-isotopic algebra", and denoted* $\hat{SL}(n,\hat{C})$. *The subset of all antisymmetric n×n real matrices X,* $X^t = -X$, *is also closed under isocommutators, is called the "isoorthogonal algebra", and is denoted* $\hat{o}(n)$.

By proceeding along similar lines, one can classify all classical, non-exceptional, Lie-Santilli isoalgebras into the isotopes of the conventional forms, denoted with $\hat{A}_n$, $\hat{B}_n$, $\hat{C}_n$ and $\hat{D}_n$ according to the general rules [25]

$$\begin{aligned}
&\textbf{Class } \hat{A}_{n-1} = \hat{SL}(n, \hat{C});\\
&\textbf{Class } \hat{B}_n = \hat{O}(2n+1, \hat{C});\\
&\textbf{Class } \hat{C}_n = \hat{SP}(n, \hat{C}); \text{ and}\\
&\textbf{Class } \hat{D}_n = \hat{O}(2n, \hat{C}).
\end{aligned} \tag{3.39}$$

plus the *isoexceptional algebras* here ignored for brevity.

Each one of the above isoalgebras then needs its own classification (evidently absent in the conventional case), depending on whether $\hat{1}$ is positive-definite (Class I), negative definite (Class II), indefinite (Class III), singular (IV) and general (Class V), as well as whether of isocharacteristic zero or p, thus illustrating the richness of the isotopic theory indicated above.

The notions of *homomorphism, automorphism and isomorphism* of two Lie-isotopic algebras $\hat{L}$ and $\hat{L}'$ are the conventional ones. Similarly, all properties of Lie algebras based on the addition, such as the *direct* and *semidirect sums*, carry over to the isotopic context unchanged (because of the preservation of the conventional additive unit 0).

By following Kadeisvili [25] we now introduce an *isoderivation* $\hat{D}$ of a Lie-isotopic algebra $\hat{L}$ as an isolinear map of $\hat{L}$ into itself satisfying the property

$$\hat{D}([\hat{A} \hat{,} \hat{B}]) = [\hat{D}(\hat{A}) \hat{,} \hat{B}] + [\hat{A} \hat{,} \hat{D}(\hat{B})] \quad \forall \hat{A}, \hat{B} \in \hat{L} . \tag{3.40}$$

If two maps $\hat{D}_1$ and $\hat{D}_2$ are isoderivations, then $\hat{a}\hat{\times}\hat{D}_1 + \hat{b}\hat{\times}\hat{D}_2$ is also an isoderivation, and the isocommutators of $\hat{D}_1$ and $\hat{D}_2$ is also an isoderivation. Thus, the set of all isoderivations forms a Lie-isotopic algebra as in the conventional case.

The isolinear map $\hat{ad}(\hat{L})$ of $\hat{L}$ into itself defined by

$$\text{Isoad } \hat{A}(\hat{B}) = [\hat{A} \hat{,} \hat{B}], \quad \forall \hat{A}, \hat{B} \in \hat{L}, \tag{3.41}$$

is called the *isoadjoint map*. It is an isoderivation, as one can prove via the Jacobi identity (3.41). The set of all $\hat{ad}(A)$ is therefore an isolinear Lie-isotopic algebra, called *isoadjoint algebra* and denoted $\hat{L}_a$. It also results to be an isoideal of the algebra of all isoderivations as in the conventional case.

Consider the isoalgebras

$$\hat{L}^{(o)} = \hat{L}, \quad \hat{L}^{(1)} = [\hat{L}^{(o)} \hat{,} \hat{L}^{(o)}], \quad \hat{L}^{(2)} = [\hat{L}^{(1)} \hat{,} \hat{L}^{(1)}], \quad \text{etc.}, \tag{3.42}$$

which are also isoideals of $\hat{L}$. $\hat{L}$ is then called *isosolvable* if, for some positive integer n, $\hat{L}^{(n)} \equiv 0$.

Consider also the sequence

$$L_{(o)} = L, \quad \hat{L}_{(1)} = [\hat{L}_{(o)} \hat{,} \hat{L}], \quad \hat{L}_{(2)} = [\hat{L}_{(1)} \hat{,} \hat{L}], \text{ etc.} \tag{3.43}$$

Then $\hat{L}$ is said to be *isonilpotent* if, for some positive integer n, $\hat{L}_{(n)} \equiv 0$. One can then see that, as in the conventional case, an isonilpotent algebra is also isosolvable, but the converse is not necessarily true.

Let the *isotrace* of a matrix be given by the element of the isofield

$$\text{Isotr } A = (\text{Tr } A)\hat{1} \in \hat{F}, \tag{3.44}$$

where TrA is the conventional trace. Then

$$\text{Isotr}\,(A \hat{\times} B) = (\text{Isotr}\,A) \hat{\times} (\text{Isotr}\,B), \qquad \text{Isotr}\,(B \hat{\times} A \hat{\times} B^{-\hat{1}}) = \text{Isotr}\,A\,. \tag{3.45}$$

Thus, Isotr A preserves the axioms of Tr A, by therefore being a correct isotopy.

Then, the isoscalar product

$$(\hat{A}\,\hat{,}\,\hat{B}) = \text{Isotr}\,[\,(\text{Isoad}\,\hat{A}) \hat{\times} (\text{Isoad}\,\hat{B})\,] \tag{3.45}$$

is called the *isokilling form* as first studied by Kadeisvili [25]. It is easy to see that $(\hat{A}\,\hat{,}\,\hat{B})$ is symmetric, bilinear, and verifies the property

$$(\text{Isoad}\,\hat{X}\,(\hat{Y})\,\hat{,}\,\hat{Z}) + (\hat{Y}\,\hat{,}\,\text{Isoad}\,\hat{X}\,(\hat{Z}) = 0, \tag{3.46}$$

thus being a correct, axiom-preserving isotopy of the conventional Killing form.

Let e_k, $k = 1, 2, ..., N$, be the basis of **L** with one-to-one invertible map $e_k \rightarrow \hat{e}_k$ into the basis $\hat{e}_k$ of $\hat{\mathbf{L}}$. Generic elements in $\hat{\mathbf{L}}$ can then be written in terms of local coordinates x, y, z,

$$\hat{A} = x^i\hat{e}_i\,,\ \hat{B} = y^j\hat{e}_j\,,\ C = z^k\hat{e}_k = [\,\hat{A}\,\hat{,}\,\hat{B}\,] = x^i y^j [\,\hat{e}_i\,\hat{,}\,\hat{e}_j\,] =$$

$$= x^i x^j\,\hat{C}_{ij}{}^k\,\hat{e}_k\,. \tag{3.47}$$

Thus,

$$[\,\text{Isoad}\,\hat{A}\,(\hat{B})\,]^k = [\,\hat{A}\,\hat{,}\,\hat{B}\,]^k = \hat{C}_{ij}{}^k\,x^i\,x^j\,. \tag{3.48}$$

We now introduce the *isocartan tensor* $\tilde{g}_{ij}$ of a Lie-isotopic algebra $\hat{\mathbf{L}}$ via the definition $(\hat{A}\,\hat{,}\,\hat{B}) = \tilde{g}_{ij}\,x^i\,y^j$ yielding

$$\tilde{g}_{ij}(t, x, \dot{x}, \ddot{x}, ...) = \hat{C}_{ip}{}^k\,\hat{C}_{jk}{}^p\,. \tag{3.49}$$

Note that the isocartan tensor has the general dependence of the isometric tensor of Sect. 2.4, thus confirming the mutual consistency among the various branches of the isotopic theory. In particular, the isocartan tensor is generally *nonlinear, nonlocal and noncanonical* in all variables x, ẋ, ẍ, ...

The isocartan tensor also clarifies another important point of the preceding analysis, that the isotopies naturally lead to an arbitrary dependence in the velocities and accelerations, and their restriction to the nonlinear dependence on the coordinates x is manifestly un-necessary.

The isotopies of the structure theory of Lie algebras then follow, including the notion of *simplicity, semisimplicity*, etc. (see the monograph [24]) Here we limit

ourselves to recall the following

Definition 3.3 [25]: *A Lie-isotopic algebra $\hat{L}$ is called "compact" ("noncompact") when the isocartan form is positive- (negative-) definite.*

Numerous additional, more refined definitions of compactness and noncompactness are possible via the isotopies of the corresponding conventional definitions [15], [126] but their study is left to interested mathematicians.

We now study a few implications of the isotopic lifting of Lie's theory.

Lemma 3.1 [52], [110]: *The isotopes of Class III $\hat{L}$ of a compact (noncompact) Lie algebra L are not necessarily compact (noncompact).*

The identification of the remaining properties which are not preserved under liftings of Class III is an instructive task for the reader. For instance, if the original structure is irreducible, its isotopic image is not necessarily so even for Class I, trivially, because the isounit itself can be reducible, thus yielding a reducible isotopic structure.

Definition 3.4 [62]: *Let $\hat{L}$ be a Lie-isotopic algebra with generators $\hat{X}_k$ and isounit $\hat{1} = \hat{T}^{-1} > 0$. The "isodual Lie-isotopic algebras" $\hat{L}^d$ is the isoalgebra with isodual generators $\hat{X}_k{}^d = -\hat{X}_k$ conventional structure functions over the isodual isofield $\hat{F}^d(\hat{a}^d,+,\hat{\times}^d)$ with "isodual isocommutators"*

$$[\hat{X}_i \hat{,} \hat{X}_j]^d = -[\hat{X}^d{}_i \hat{,} \hat{X}^d{}_j] = -[\hat{X}_i \hat{,} \hat{X}_j] = \hat{C}_{ij}{}^k \hat{X}^d{}_k = -\hat{C}_{ij}{}^k \hat{X}_k. \qquad (3.50)$$

When the original algebra is a Lie algebra L the "isodual Lie algebra" is given by the structure L^d over the isodual field $F^d(a^d,+,\times^d)$ with "isodual commutators"

$$[\hat{X}_i , \hat{X}_j]^d = \hat{X}_i \times^d \hat{X}_j - \hat{X}_j \times^d \hat{X}_i = -[\hat{X}_i , \hat{X}_j] = -C_{ij}{}^k \hat{X}_k. \qquad (3.51)$$

$\hat{L}$ and $\hat{L}^d$ are then anti-isomorphic. Note that the isoalgebras of Class III contain the isoalgebras $\hat{L}$ and the isoduals $\hat{L}^d$. The above remarks therefore show that the Lie-isotopic theory can be naturally formulated for Class III, as implicitly done above.

Note the necessity of the isotopies for the very construction of the isodual of conventional Lie algebras. In fact, they require the nontrivial lift of the unit $I \rightarrow I^d = (-I)$, with consequential necessary generalization of the Lie product $AB - BA$ into the isotopic form $A{\times}T{\times}B - B{\times}T{\times}A$.

The following property is mathematically simple, yet carries important physical applications.

Theorem 3.6 [52], [115]: *All infinitely possible, isotopes $\hat{\mathbf{L}}$ of Class I of a (finite-dimensional) Lie algebra* $\mathbf{L}$ *are locally isomorphic to* $\mathbf{L}$, *and all infinitely possible isodual isotopes* $\hat{\mathbf{L}}^d$ *of Class II are locally isomorphic to* $\mathbf{L}^d$

The simplest possible proof is via the redefinition of the basis $\hat{X}_k \rightarrow \hat{X}'_k = X_k \hat{1}$, under which isotopic algebras $\hat{\mathbf{L}}$ acquire the same structure constants of $\mathbf{L}$,

$$[\hat{X}_i \hat{,} \hat{X}_j] \rightarrow [\hat{X}'_i \hat{,} \hat{X}'_j] = [\hat{X}_i , \hat{X}_j]\hat{1} = C_{ij}^{\ k} \hat{X}'_k . \tag{3.52}$$

We should however indicate that, even though the above reduction is possible, in general we have $\hat{C}_{ij}^{\ k} \neq C_{ij}^{\ k} \hat{1}$, thus rendering inapplicable the realization $\hat{X}' = X\,\hat{1}$. Also the realization $\hat{X}'_k = X_k \hat{1}$ does not yield the desired nonlinear–nonlocal–nonhamiltonian isosymmetries as we shall see in Sect. 4.6.

Despite the local isomorphism $\mathbf{L} \approx \hat{\mathbf{L}}$, the lifting $\mathbf{L} \rightarrow \hat{\mathbf{L}}$ is not mathematically trivial because these two algebras are not unitarily equivalent. The physical relevance of the isotopies originates precisely from their local isomorphism, because it permits the construction of nonlinear, nonlocal and noncanonical isotopes of the rotational $\hat{SO}(3)$, Galilean $\hat{G}(3.1)$, Lorentz $\hat{O}(3.1)$, Poincaré $\hat{P}(3.1)$, $\hat{SU}(3)$ and other space–time and internal symmetries which are locally isomorphic to the original algebras [116].

We now illustrate the results of this sections with the isotopies and isodualities of the rotational algebra **so(3)** with generators in their adjoint form (3.16)). For this purpose, the isounit and isotopic element of Class III, can be realized in the form

$$\hat{1} = \mathrm{diag.}\,(\pm b_1^{-2}, \pm b_2^{-2}, \pm b_3^{-2}), \quad b_k(t, r, \dot{r}, \ddot{r}, ...) \neq 0,$$

$$\delta = \hat{T} = \mathrm{diag.}\,(\pm b_1^{2}, \pm b_2^{2}, \pm b_3^{2}), \tag{3.53}$$

The Isotopic Second Theorem 3.2 then yields the isocommutation rules

$$[\mathfrak{J}_i \hat{,} \mathfrak{J}_j] = \mathfrak{J}_i \times \hat{T} \times \mathfrak{J}_j - \mathfrak{J}_j \times \hat{T} \times \mathfrak{J}_i = \hat{C}_{ij}^{\ k}(t, r, \dot{r}, \ddot{r}, ...) \times \hat{T} \times \mathfrak{J}_k , \tag{3.54}$$

where the $\mathfrak{J}$'s are the conventional adjoint generators and the $\hat{C}$'s are the structure functions.

It is easy to see that all possible isoalgebras (3.54) are given by [63]:

1) **so(3)** for $\hat{T} \equiv I = \mathrm{diag.}\,(1, 1, 1)$ with commutation rules

$$[J_1, J_2] = J_3, \ [J_2, J_3] = J_1, \ [J_3, J_1] = J_2; \tag{355)}$$

2) **so(2.1)** for $\hat{T} = \text{diag.}(1, -1, 1)$ with rules

$$[\hat{J}_1 \hat{,} \hat{J}_2] = \hat{J}_3, \ [\hat{J}_2 \hat{,} \hat{J}_3] = -\hat{J}_1, \ [\hat{J}_3 \hat{,} \hat{J}_1] = \hat{J}_2; \tag{3.56}$$

3) An infinite family of isotopes **$\hat{so}$(3)** isomorphic to so(3) for $\hat{T} = \text{diag.}(b_1^2, b_2^2, b_3^2)$ with rules

$$[\hat{J}_1 \hat{,} \hat{J}_2] = b_3^2 \hat{J}_3, \ [\hat{J}_2 \hat{,} \hat{J}_3] = b_1^2 \hat{J}_1, \ [\hat{J}_3 \hat{,} \hat{J}_1] = b_2^2 \hat{J}_2; \tag{3.57}$$

4) An infinite family of isotopes **$\hat{so}$(2.1)** isomorphic to so(2.1) for $\hat{T} = \text{diag.}(b_1^2, -b_2^2, b_3^2)$ and rules

$$[\hat{J}_1 \hat{,} \hat{J}_2] = b_3^2 \hat{J}_3, \ [\hat{J}_2 \hat{,} \hat{J}_3] = -b_1^2 \hat{J}_1, \ [\hat{J}_3 \hat{,} \hat{J}_1] = b_2^2 \hat{J}_2; \tag{3.58}$$

5) The isodual **so^d(3)** of **so(3)** for $\hat{T} = \text{diag.}(-1, -1, -1)$ and rules

$$[\hat{J}_1 \hat{,} \hat{J}_2] = -\hat{J}_3, \ [\hat{J}_2 \hat{,} \hat{J}_3] = -\hat{J}_1, \ [\hat{J}_3 \hat{,} \hat{J}_1] = -\hat{J}_2; \tag{3.59}$$

6) The isodual so^d(2.1) of **so(2.1)** for $\hat{T} = \text{diag.}(-1, 1, -1)$ and rules

$$[\hat{J}_1 \hat{,} \hat{J}_2] = -\hat{J}_3, \ [\hat{J}_2 \hat{,} \hat{J}_3] = \hat{J}_1, \ [\hat{J}_3 \hat{,} \hat{J}_1] = -\hat{J}_2; \tag{3.60}$$

7) The infinite family of isotopes **$\hat{so}^d(3) \approx so^d(3)$** for $\hat{T} = \text{diag.}(-b_1^2, -b_2^2, -b_3^2)$ and rules

$$[\hat{J}_1 \hat{,} \hat{J}_2] = -b_3^2 \hat{J}_3, \ [\hat{J}_2 \hat{,} \hat{J}_3] = -b_1^2 \hat{J}_1, \ [\hat{J}_3 \hat{,} \hat{J}_1] = -b_2^2 \hat{J}_2; \tag{3.61}$$

8) The infinite family of isotopes **$\hat{so}^d(2.1) \approx SO^d(2.1)$** for $\hat{T} = \text{diag.}(-b_1^2, b_2^2, -b_3^2)$ and rules

$$[\hat{J}_1 \hat{,} \hat{J}_2] = -b_3^2 \hat{J}_3, \ [\hat{J}_2 \hat{,} \hat{J}_3] = b_1^2 \hat{J}_1, \ [\hat{J}_3 \hat{,} \hat{J}_1] = -b_2^2 \hat{J}_2; \tag{3.62}$$

The reader can readily verify the above indicated local isomorphisms via the redefinition of the basis

$$\hat{J}'_1 = b_1^{-1} b_3^{-1} \hat{J}_1, \quad \hat{J}'_2 = b_1^{-1} b_3^{-1} \hat{J}_2, \quad \hat{J}'_3 = b_1^{-1} b_2^{-1} \hat{J}_3, \tag{3.63}$$

in which case the b-terms in the r.h.s. of the commutation rules disappear and one

recovers conventional structure constants of so(3) and so(2.1) under isotopies (see Ch. II–6 for details).

It is instructive for the interested reader to verify with the above examples various other notions introduced in this section, such as the isocartan's tensor, the isokilling form, etc.

The *classical realization* of the Lie–Santilli isoalgebra is that via vector isofields X_i on the isocotangent bundle (Sect. 2.8), while the *operator realization* is that of Sect. 2.6.

It is significant to recall that Santilli presented his isotopies of Lie's Theorem as *particular cases* of the brooder genotopic formulation of the same theorems with a Lie–admissible structure, which are omitted here for brevity. We merely limit ourselves to indicate that the isotopic product is lifted into the *genotopic product*

$$(A, B) = A < B = B > A = A \times \hat{R} \times B - B \times \hat{S} \times A, \tag{3.64}$$

which verify the axioms for the third definition of Lie–admissibility of Sect. 1.3.

In an informal seminar at the *International Congress of Mathematicians*, held in Zurich, Switzerland, on August 1994, Santilli pointed out that *the product* (A, B) *is Lie–admissible if computed in conventional spaces over conventional fields, while the same product verifies the Lie axioms when each term of the product is computed in its appropriate genospaces and genofields.*

In fact, the genoproduct exhibits in a natural way the ordering to the left and that to the right with consequential origin from the corresponding genoenvelopes (Sect. .3.2), and can be written (see next section for details)

$$(A, B) = {}_{{}^{<}\xi|} A < B - B > A_{|\xi^{>}} . \tag{3.65}$$

In this case the product (A, B) verifies the *Lie* rather than the *Lie–admissible* axioms because the genoenvelope ${}^{<}\xi$ and related genoproduct A<B, when referred to its own genounits ${}^{<}1$ is isomorphic to the conventional envelope ξ with product A×B and unit 1, and the same happens for the conjugate genoenvelope $\xi^{>}$.

Equivalently, we can say that the genotopy $A\times B \rightarrow A<B = A\times\hat{R}\times B$ is an isomorphism when A×B is referred to the unit 1 and A<B is referred to the unit ${}^{<}1 = \hat{R}^{-1}$, and the same happed for the conjugate genotopy $A\times B \rightarrow A>B = A\times\hat{S}\times B$.

The *Lie–Santilli hyperalgebras* can be defined via the hyperproduct

$$\{A, B\} = \{A\}\{<\}\{B\} - \{B\}\{>\}\{A\} \tag{3.66}$$

where we have used the symbols of Sect. 2.2.

It is intriguing to note that, when each ordered multivalued product is referred to its appropriate hyperenvelope and related hyperfield, the above hyperproduct too satisfies the *Lie* axioms, thus permitting a further structural broadening of the Lie-Santilli iso- and geno-theories.

II.3.4 Lie-Santilli isogroups and their isoduals

The isotopies of a topological groups are still lacking at this writing. Only the isotopies of Lie's transformation groups are available and they can formulated via the following

Definition 3.5 [52], [115]: *A "right Lie-Santilli (transformation) isogroup" $\hat{G}$, or "isogroup" for short, on an isospace $\hat{S}(\hat{x},\hat{F})$ over an isofield $\hat{F}(\hat{a},+,\hat{\times})$ (of isoreal numbers $\hat{R}$ or isocomplex numbers $\hat{C}$ or isoquaternions $\hat{Q}$) is a group which maps each element $\hat{x} \in \hat{S}(\hat{x},\hat{F})$ into a new element $\hat{x}' \in \hat{S}(\hat{x},\hat{F})$ via the isotransformations*

$$\hat{x}' = \hat{U} \hat{\times} \hat{x} = \hat{U} \times \hat{T} \times \hat{x}, \qquad \hat{T} \ \textit{fixed}, \tag{3.67}$$

such that:
1) The map $(\hat{U}, \hat{x}) \rightarrow \hat{U} \hat{\times} \hat{x}$ *of* $\hat{G}\hat{\times}\hat{S}(\hat{x},\hat{F})$ *onto* $\hat{S}(\hat{x},\hat{F})$ *is differentiable;*
2) $\hat{1} \hat{\times} \hat{U} = \hat{U} \hat{\times} \hat{1} = \hat{U}, \ \forall \ \hat{U} \in \hat{G}$; and
3) $\hat{U}_1 \hat{\times} (\hat{U}_2 \hat{\times} \hat{x}) = (\hat{U}_1 \hat{\times} \hat{U}_2) \hat{\times} \hat{x}, \ \forall \ \hat{x} \in \hat{S}(\hat{x},\hat{F})$ *and* $\hat{U}_1, \hat{U}_2 \in \hat{G}$.
A "left Lie-Santilli (transformation) isogroup" is defined accordingly.

Right or left isogroups are characterized by the following *isogroup laws* first introduced in ref. [52]

$$\hat{U}(\hat{0}) = \hat{1}, \quad \hat{U}(\hat{w}) \hat{\times} \hat{U}(\hat{w}') = \hat{U}(\hat{w}') \hat{\times} \hat{U}(\hat{w}) = \hat{U}(\hat{w} + \hat{w}'), \quad \hat{U}(\hat{w}) \hat{\times} \hat{U}(-\hat{w}) = \hat{1}, \qquad \hat{w} \in \hat{F}, \tag{3.68}$$

A significant function of the isogroups is that of identifying the group structure of the classical and operator time evolution of isotopic theories, according to the isotransforms

$$\hat{x}' = \hat{U}(t) \hat{\times} \hat{x}, \quad \hat{U} = (\hat{e}^{i\hat{H} \hat{\times} \hat{t}}) \hat{\times} x = (e^{iH \times \hat{T} \times t}) \times x \tag{3.69}$$

where we have used the isoexponentiation (3.10). which do indeed constitute Lie-Santilli isogroups as per Definition 3.5.

Note the insufficiency of the conventional Lie groups $x' = e^{H \times t} x$ for the characterization of structures (3.69) on numerous independent grounds, such as: Lie (transformation) groups have a linear, local and canonical structure while structures (3.69) are nonlinear, nonlocal and noncanonical; Lie groups are dependent on the form $I = \text{diag.}(1, 1, \dots, 1)$ of the basic unit, while structures (3.69) have arbitrary integro-differential quantities $\hat{1}$ for basic unit; etc.

Most of the studies conducted on isotopies until now have been focused on the achievement of a formulation of functional analysis, geometries and mechanics compatible with the isotopic structure of groups (3.69).

The notions of *connected* or *simply connected transformation groups* (see, e.g., ref.s [15,], [126]) carry over to the Lie-isotopic groups in their entirety. We consider hereon the connected isotransformation groups (see Sect. 3.8 for the discrete part).

An important property permitting the isocomposition of Lie-isotopic groups is given by the following

Theorem 3.7 (Baker-Campbell-Hausdorff–Santilli Theorem [52], [110]): *The conventional group composition laws admit a consistent isotopic lifting, resulting in the following "isotopic composition law"*

$$\hat{U}_1 \hat{\times} \hat{U}_2 = \{ e_{\xi}^{\hat{X}_1} \} \hat{\times} \{ e_{\xi}^{\hat{X}_2} \} = \hat{U}_3 = e_{\xi}^{\hat{X}_3},$$

$$\hat{X}_3 = \hat{X}_1 + \hat{X}_2 + [\hat{X}_1 \hat{,} \hat{X}_2] / 2 + [(\hat{X}_1 - \hat{X}_2) \hat{,} [\hat{X}_1 \hat{,} \hat{X}_2]] / 12 + \dots \qquad (3.70)$$

By following Kadeisvili [25], we now study the connection between Lie-Santilli isogroups and isoalgebras. Let $\hat{L}$ be a (finite-dimensional) Lie-isotopic algebra with (ordered) basis $\hat{X}_k$, $k = 1, 2, \dots, N$. For a sufficiently small neighborhood N of the isoorigin of $\hat{L}$, a generic element of $\hat{G}$ can be written

$$\hat{U}(\hat{w}) = \hat{\prod}_{k=1,2,\dots,N} e_{\xi}^{i \hat{X}_k \hat{\times} \hat{w}_k}. \qquad (3.71)$$

which characterizes some open neighborhood $\hat{M}$ of the isounit $\hat{1}$ of $\hat{G}$.

The map

$$\Phi_{\hat{U}_1}(\hat{U}_2) = \hat{U}_1 \hat{\times} \hat{U}_2 \hat{\times} \hat{U}_1^{-1}, \qquad (3.72)$$

for a fixed $\hat{U}_1 \in \hat{G}$, characterizes an *inner isoautomorphism* of $\hat{G}$ onto itself. The corresponding isoautomorphism of the algebra $\hat{L}$ can be readily computed by considering expression (4.5.7) in the neighborhood of the isounit $\hat{1}$, in which case we have

$$\hat{U}'_2 = \hat{U}_1 \hat{\times} \hat{U}_2 \hat{\times} \hat{U}_1^{-\hat{1}} \cong \hat{U}_2 + \hat{w}_1 \hat{\times} \hat{w}_2 \hat{\times} [\hat{X}_2 \hat{,} \hat{X}_1] + O^{(2)} . \tag{3.73}$$

By recalling the differentiability property of $\hat{G}$, we also have the following isotopy of the conventional expression in one dimension

$$(1/i) \frac{d}{dw} \hat{U}_{|\hat{w}=0} = (1/i) \frac{d}{dw} e_{\xi}^{iwX}{}_{|w=0} = \hat{X} \hat{\times} e_{\xi}^{iwX}{}_{|w=0} = \hat{X} , \tag{3.74}$$

Thus, to every inner isoautomorphism of $\hat{G}$ there corresponds an inner isoautomorphism of $\hat{L}$ which can be expressed in the form [25]

$$(\hat{L})_i{}^j = \hat{C}_{ki}{}^j w^k . \tag{3.75}$$

The Lie-isotopic group $\hat{G}_a$ of all inner isoautomorphism of $\hat{G}$ is called the *isoadjoint group*. It is possible to prove that the Lie-isotopic algebra of $\hat{G}_a$ is the isoadjoint algebra $\hat{L}_a$ of $\hat{L}$.

We mentioned before that the direct sum of Lie-isotopic algebras is the conventional operation because the addition is not lifted in our studies. The corresponding operation for groups is the semidirect product which, as such, demands care in its formulation.

Let $\hat{G}$ be a Lie-isotopic group and $\hat{G}_a$ the group of all its inner isoautomorphisms. Let $\hat{G}^o{}_a$ be a subgroup of $\hat{G}_a$, and let $\hat{\Lambda}(\hat{g})$ be the image of $\hat{g} \in \hat{G}$ under $\hat{G}^o{}_a$. The *semidirect isoproduct* $\hat{G} \hat{\times} \hat{G}^o{}_a$ of $\hat{G}$ and $\hat{G}^o{}_a$ is the Lie-isotopic group of all ordered pairs $(\hat{g}, \hat{\Lambda})$ with group isomultiplication

$$(\hat{g}, \hat{\Lambda}) \hat{\times} (\hat{g}', \hat{\Lambda}') = (\hat{g} \hat{\times} \hat{\Lambda}(\hat{g}'), \hat{\Lambda} \hat{\times} \hat{\Lambda}') . \tag{3.76}$$

with total isounit given by

$$\hat{1}_{tot} = (\hat{1}, \hat{1}_{\hat{\Lambda}}) , \tag{3.77}$$

and inverse

$$(\hat{g}, \hat{\Lambda})^{-\hat{1}} = (\hat{\Lambda}^{-\hat{1}}(\hat{g}^{-\hat{1}}), \hat{\Lambda}^{-\hat{1}}) . \tag{3.78}$$

The above notions play an important role in the isotopies of the inhomogeneous space-time symmetries, such as Galilei's and Poincaré's symmetries (Sect. 3.9).

Let $\hat{G}_1$ and $\hat{G}_2$ be two Lie-isotopic groups with respective isounits $\hat{1}_1$ and $\hat{1}_2$. The *direct isoproduct* $\hat{G}_1 \hat{\otimes} \hat{G}_2$ of $\hat{G}_1$ and $\hat{G}_2$ is the Lie-isotopic group of all ordered pairs $g = (\hat{g}_1, \hat{g}_2)$, $\hat{g}_1 \in \hat{G}_1$, $\hat{g}_2 \in \hat{G}_2$, with isomultiplication

$$g \hat{\times} g' = (\hat{g}_1, \hat{g}_2) \hat{\times} (\hat{g}'_1, \hat{g}'_2) = (\hat{g}_1 \hat{\times} \hat{g}'_1, \hat{g}_2 \hat{\times} \hat{g}'_2), \tag{3.79}$$

total isounit

$$\hat{1}_{tot} = (\hat{1}_1, \hat{1}_2), \tag{3.80}$$

and inverse

$$g^{-1} = (\hat{g}_1^{-1}, \hat{g}_2^{-1}). \tag{3.81}$$

Definition 3.6 [62], [115]:: *Let $\hat{G}$ be an N-dimensional isotransformation group of Class I with infinitesimal generators $\hat{X}_k$, $k = 1, 2, ..., N$. The "isodual image" $\hat{G}^d$ of $\hat{G}$ is the N-dimensional isogroup with infinitesimal generators $\hat{X}^d{}_k = -\hat{X}_k$, isodual isounit $\hat{1}^d = -\hat{1}$ and isodual parameters $\hat{w}^d = -\hat{w}$ over the isodual isofield $\hat{F}^d(\hat{a}^d,+,\hat{\times}^d)$ with "isodual isotransformation" in a suitable neighborhood of $\hat{1}^d$*

$$\hat{x}^{d\,\prime} = \hat{U}^d(\hat{w}^d) \hat{\times}^d \hat{x}^d = \{ e_{\xi^d}{}^{i^d \hat{X}^d \hat{\times}^d \hat{w}^d} \} \hat{\times}^d \hat{x}^d = -\{ e_{\xi}{}^{i X T w} \} \hat{x}^d. \tag{3.82}$$

In particular, the above antiautomorphic conjugation can also be defined for conventional Lie groups, yielding the "isodual Lie group" G^d which is defined over the isodual field $F^d(a^d,+,\times^d)$ with generic "isodual transformations"

$$x^{d\,\prime} = U^d(w^d) \times^d x^d = \{ \hat{e}^{d\, i X^d \times^d w^d} \} \times^d x^d = -\{ e^{i X w} \} x^d. \tag{3.83}$$

In summary, an abstract Lie group admits the following four realizations relevant for our analysis:

Lie groups G of conventional type;
Lie-Santilli isogroups $\hat{G}$;
Isodual Lie groups G^d; and
Isodual Lie-Santilli isogroups $\hat{G}^d$. (3.84)

Realization G (G^d) is useful for the characterization of particles (antiparticles) in the homogeneous and isotropic vacuum, while realization $\hat{G}$ ($\hat{G}^d$) is useful for the characterization of particles (antiparticles) within inhomogeneous and anisotropic physical media.

It is hoped the reader can see from the above foundations that the entire conventional Lie's theory does indeed admit a consistent and nontrivial isotopic covering.

A classical realization of the Lie–Santilli isogroups can be formulated on the isotangent bundle $T^*\hat{E}(r,\hat{\delta},\hat{R})$, $\hat{\delta} = T\delta$, with local chart $a = \{r^k, p_k\}$, $\mu = 1, 2, 3, 4, 5, 6$, $k = 1, 2, 3$, and isounit

$$\hat{1}_2 = \mathrm{diag.}\,(\hat{1}, \hat{T}). \tag{3.85}$$

The Hamilton–Santilli equations (2.53), i.e.,

$$\partial a^\mu / \partial t = \hat{\Xi}^\mu = \omega^{\mu\alpha}\, T_{2\alpha}{}^{\nu} \frac{\partial H}{\partial a^\nu}, \tag{3.86}$$

where $\omega^{\mu\alpha}$ is the familiar canonical Lie tensor. Eq.s (2.58) can be isoexponentiated and, after factorization of the isounit, can be written

$$a(t) = \{\, \hat{e}^{t\hat{\Xi}\partial/\partial a^\mu} \} * a(0) = \{\, e^{t\omega^{\mu\alpha} T_{2\alpha}{}^{\nu} (\partial H/\partial a^\mu)\, \partial/\partial a^\alpha} \}\, a(0), \tag{3.87}$$

where we have ignored the factorization of the isounit in the isoexponent for simplicity. The computation of the Lie–santilli isoalgebra is consequential and coincides at the abstract level with the conventional formulation in terms of vector fields.

An operator realization of the Lie–Santilli isogroups is given by *isounitary transformations* $x' = \hat{U} \times x$ on an isohilbert space $\mathcal{H}$ [100] with

$$\hat{U} \hat{\times} \hat{U}^\dagger = \hat{U}^\dagger \hat{\times} \hat{U} = \hat{1}, \tag{3.88}$$

with action on an observable Q realization via an *isohermitean operator* H according to Eq. (2.79) which can now be written in the full isotopic form thanks to Theorem 3.1,

$$Q(t) = (\, \hat{e}^{\,iH\times t}) \hat{\times} Q(t) \hat{\times} (\, \hat{e}^{\,-it\times H}) = (\, e^{\,iH\times\hat{T}\times t}) \times Q(0) \times (\, e^{\,-it\times\hat{T}\times H}), \tag{3.89}$$

The use of the bimodular isotransforms and the techniques studied in this section, then characterize the corresponding Lie–Santilli isoalgebra expressed via infinitesimal time evolution law (2.78), thus confirming the interconnection and mutual compatibility between isoalgebras and isogroups in exactly the same manner as that for the conventional theory.

The above classical and operator realizations are also interconnected in a

unique and unambiguous way by the isoquantization [100].

It is easy to see that isogroup (3.89) has a *natural ordering to the right and to the left*, thus requiring in a natural way two different isoenvelopes, one for the action to the right and the other for the action to the left, interconnected with Hermitean conjugation.

This is precisely the structure of the *Lie–Santilli genogroups* which can be written

$$Q(t) = (\hat{e}^{>\, iH\times t}) > Q(t) < ({}^{<}\hat{e}^{\,-it\times H}) = (e^{\,iH\times \hat{S}\times t}) \times Q(0) \times (e^{-it\times \hat{R}\times H}), \qquad (3.89)$$

and which yields as infinitesimal form the Lie–admissible time evolution (2.84), with product (3.3.65).

Finally, we can formulate the *hypergroup* according to the structure

$$\{Q(t)\} = (\{\hat{e}^{>}\}^{\, i\{H\}\times t}) \{>\} \{Q(t)\} \{<\} (\{{}^{<}\hat{e}\}^{\,-it\times \{H\}}), \qquad (3.90)$$

where we have used the same notation as in preceding sections, which yields as infinitesimal version the hyperproduct (3.66).

The latter properties are sufficient to indicate the possibility that the the isotheory admits a step–by–step further generalization of genotopic and structural type.

II.3.5. Santilli's fundamental theorem on isosymmetries.

One of the most important application of Lie's theory is that for the construction of the *symmetries* of linear, local–differential and canonical systems. Along the same lines, one of the most important application of the Lie–Santilli isotheory is that for the construction of symmetries, this time, of nonlinear, nonlocal and noncanonical systems. The latter objective is embodied in the following important property which we quote for brevity without proof:

Theorem 3.7 [62]. *Let* G *be an N–dimensional Lie symmetry group of an m–dimensional metric or pseudo–metric space* S(x,g,F) *over a field* F

$$G: \; x' = A(w)\; x, \quad (x'-y')^{t}\, A^{\dagger}\, g\, A\, (x-y) \equiv (x-y)^{\dagger}\, g\, (x-y),$$

$$A^{\dagger}\, g\, A \; = \; A\, g\, A^{\dagger} \; = \; g\, . \qquad (3.91)$$

where t *stands for transpose and* † *for Hermitean conjugation. Then the infinitely possible isotopes* $\hat{G}$ *of* G *of Class* III *characterized by the same generators and parameters of* G *and new isounits* $\hat{1}$ *(isotopic elements* T), *automatically leave invariant the iso-composition on the isospaces* $\hat{S}(x,\hat{g},\hat{F})$, $\hat{g} = Tg$, $\hat{1} = T^{-1}$,

$$\hat{G}: x' = \hat{A}(w) \hat{\times} x, \; (x'-y')^t \hat{\times} \hat{A}^\dagger \hat{g} \hat{A} \hat{\times} (x-y) =$$

$$= (x-y)^t \hat{g} (x-y), \quad \hat{A}^\dagger \hat{g} \hat{A} = \hat{A} \hat{g} \hat{A}^\dagger = \hat{1} \hat{g} \hat{1}, \tag{3.92}$$

The "direct universal" of the resulting isosymmetries for all infinitely possible isotopies $g \rightarrow \hat{g} = \hat{T}(t, x, \dot{x}, \ddot{x}, ...)g$ is then evident owing to the completely unrestricted functional dependence of the isotopic element T. One should also note the *insufficiency* of the so-called *trivial isotopy*

$$X_k \rightarrow X'_k = X_k \hat{1}, \tag{3.93}$$

for the achievement of the desired form-invariance. In fact, under the above mapping the isoexponentiation becomes

$$e_{\xi}^{\, i X'_k * w_k} = \{ e^{iX'_k T w_k} \}\hat{1} = \{ e^{iX_k w_k} \}\hat{1}, \tag{3.94}$$

namely, we have the disappearance precisely of the isotopic element T in the exponent which provides the invariance of the isoseparation.

II.3.6. Isotopies and isodualities of the rotational symmetry.

One of the most important results achieved by Santilli as a culmination of all his efforts [47]-[118] is a generalization of the current formulation of the space-time symmetries of contemporary physics, the rotational O(3), Lorentz L(3.1) and Poincaré P(3.1) = L(3.1)×T(3.1) symmetries, with far reaching mathematical and physical implications.

These generalized symmetries have been solely studied by Santilli up to this writing. The isorotational symmetry was studied in papers [62], [63], the isolorentz symmetry was studied in ref. [59] of 1983; their operator image in paper [60] of the same year; a comprehensive classical study in memoir [67]; a comprehensive operator counterpart in memoir [72] of 1992 (with the first experimental verification via the Bose-Einstein correlation); a comprehensive classical and operator study in

paper [79] of 1993; specific studied on the spinorial case were conducted in paper [95] with additional experimental verifications; a detailed classical treatment in monograph [114] and the operator treatment in monograph [116].

In this section we shall study the isorotational symmetry, while the remaining isosymmetries are studied in the subsequent two sections.

Consider the lifting of the perfect sphere in Euclidean space $E(r,\delta,\Re)$ with local coordinates $r = (x, y, z)$, and metric $\delta = \mathrm{diag.}\,(1, 1, 1)$ over the reals $\Re$,

$$r^2 = r^t \delta r = xx + yy + zz, \qquad (3.95)$$

into the most general possible ellipsoid of Class III on isospace $\hat{E}^{III}(r,\hat{\delta},\hat{\Re})$, $\hat{\delta} = T\delta$, $T = \mathrm{diag.}\,(g_{11}, g_{22}, g_{33})$, $\hat{1} = T^{-1}$.

$$r^{\hat{2}} = r^t \hat{\delta} r = x g_{11} y + y g_{22} y + z g_{33} z,$$

$$\delta^\dagger = \hat{\delta}, \hat{\ } g_{kk} = g_{kk}(t, r, \dot{r}, \ddot{r}, \ldots) \neq 0, \qquad (3.96)$$

The invariance of the original separation r^2 is the conventional rotational symmetry O(3). The isotopic techniques permit the construction, in the needed explicit and finite form, of the isosymmetries $\hat{O}(3)$ of all infinitely possible generalized invariants $r^{\hat{2}}$ via the following steps: (1) Identification of the basic isotopic element T in the lifting $\delta \to \hat{\delta} = T\delta$ which, in this particular case, is given by the new metric $\hat{\delta}$ itself, $T \equiv \hat{\delta}$, and identification of the fundamental unit of the theory, $\hat{1} = T^{-1}$; (2) Consequential lifting of the basic field $\Re(n,+,\times) \Rightarrow \hat{\Re}(\hat{n},+,*)$; (3) Identification of the isospace in which the generalized metric $\hat{\delta}$ is defined, which is given by the three-dimensional isoeuclidean spaces $\hat{E}(r,\hat{\delta},\hat{\Re})$, $\hat{\delta} = T\delta$, $\hat{1} = T^{-1}$; (4) Construction of the $\hat{O}(3)$ symmetry via the use of the original parameters of O(3) (the Euler's angles θ_k, $k = 1, 2, 3$), the original generators (the angular momentum components $M_k = \epsilon_{kij}\, r^i\, p_j$) in their fundamental (adjoint) representation, and the new metric $\hat{\delta}$; and (5) Classification, interpretation and application of the results.

The explicit construction of $\hat{O}(3)$ is straightforward. According to the Lie–Santilli theory, the connected component $S\hat{O}(3)$ of $\hat{O}(3)$ is given by [63]

$$S\hat{O}(3): \quad r' = \hat{R}(\theta) \hat{\times} r, \quad \hat{R}(\theta) = \prod{}^{\hat{\times}}_{k=1,2,3}\, e_{\xi}^{\,iM_k\theta_k} =$$

$$= \{\prod\nolimits_{k=1,2,3} e^{\,iM_k T\theta_k}\} \times \hat{1}, \qquad (3.97)$$

while the discrete component is given by the *isoinversions* [loc. cit.] $r' = \hat{\pi} * r = \pi r = -r$, where π is the conventional inversion.

Under the assumed conditions on the isotopic element T, the convergence of

isoexponentiations is ensured by the original convergence, thus permitting the explicit construction of the isorotations, with example around the third axis [53]

$$x' = x\cos[\theta_3(g_{11}g_{22})^{\frac{1}{2}}] + y\, g_{22}(g_{11}g_{22})^{-\frac{1}{2}}\sin[\theta_3(g_{11}g_{22})^{\frac{1}{2}}],$$

$$y' = -x\, g_{11}(g_{11}g_{22})^{-\frac{1}{2}}\sin[\theta_3(g_{11}g_{22})^{\frac{1}{2}}] + y\cos[\theta_3[g_{11}g_{22})^{\frac{1}{2}}],$$

$$z' = z. \qquad (3.98)$$

(see [116] for general isorotations). One should note that the argument of the trigonometric functions as derived via the above isoexponentiation coincides with the isoangle of the isotrigonometry in $\hat{E}(r,\hat{\delta},\hat{\mathfrak{R}})$ (see paper [60]) thus confirming the remarkable compatibility and interconnections of the various branches of the the isotopic theory.

The computation of the isoalgebras $\hat{\mathbf{o}}(\mathbf{3})$ of $\hat{O}(3)$ is then straighforward. By assuming the M's to be in theor conventional regular representation, we have [63]

$$\hat{o}(3): [M_i \hat{,} M_j] = M_i \times T \times M_j - M_j \times T \times M_i = \hat{C}_{ij}{}^k \hat{\times} M_k, \qquad (3.99)$$

where $\hat{C}_{ij}{}^k = \epsilon_{ijk}\, g_{kk}{}^{-1}\hat{1}$. The above isoalgebra illustrates the explicit dependence of the structure *functions*. The proof of the isomorphism $\hat{o}(3) \approx o(3)$ was done [loc. cit.] via a suitable reformulation of the basis under which the structure functions recover the value $\hat{\epsilon}_{ijk} = \epsilon_{ijk}\,\hat{1}$ .The isocenter of $s\hat{o}(3)$ is characterized by the *isocasimir invariants*

$$C^{(0)} = \hat{1}, \quad C^{(2)} = M^{\hat{2}} = M \hat{\times} M = \sum_{k=1,2,3} M_k \times T \times M_k. \qquad (3.100)$$

In hadronic mechanics [116] one of the possible realizations is the following. The linear momentum operator has the isotopic form

$$p_k \hat{\times} |\hat{\psi}> = -i\,\hat{\nabla}_k |\hat{\psi}> = -i\,\hat{T}_k{}^i\, \nabla_i |\hat{\psi}>. \qquad (3.101)$$

where we have used the isodifferential calculus of Sect. 2.4. The fundamental isocommutation rules are then given by

$$[r^i \hat{,} p_j] = i\,\delta^i{}_j\,, \quad [r^i \hat{,} r^j] = [p_i \hat{,} p_j] = 0\,. \qquad (3.102)$$

Note that in their contravariant form the coordinates are given by $r_k = \hat{\delta}_{ki}\, r^i$. In this case the fundamental isocommutation rules are given by

$$[\, r_i \hat{,} p_j] = i\,\delta^i{}_j = i\,\hat{1}\,\delta^i{}_j, \quad [\, r_i \hat{,} r_j] = [p_i \hat{,} p_j] = 0\,, \tag{3.103}$$

The operator isoalgebra $\hat{o}(3)$ with generators $M_k = \epsilon_{kij}\, r^i\, p_j$ is then given by

$$\hat{o}(3): \quad [\, M_i \hat{,} M_j\,] = M_i \times T \times M_j - M_j \times T \times M_i = i\, \hat{\epsilon}_{ij}{}^k \,\hat{\times}\, M_k\,, \tag{3.104}$$

see [61] for details). The above results illustrates again the abstract identity of quantum and hadronic mechanics.

Note the nonlinear–nonlocal–noncanonical character of isotransformations (3.45) owing to the unrestricted functional dependence of the diagonal elements g_{kk}. Note also the extreme simplicity of the final results. In fact, the explicit symmetry transformations of separation (3.43) are provided by just plotting the given g_{kk} values into transformations (3.45) without any need of any additional computation. Note finally that the above invariance includes as particular case the general isosymmetry $\hat{O}(3)$ of (the space-component of) gravitation which, since it is locally Euclidean, remains isomorphic to $O(3)$.

As an example, the symmetry of the space-component of the Schwarzschild line element is given by plotting the following values

$$g_{11} = (1 - M/r)^{-1}, \quad g_{22} = r^2, \quad g_{33} = r^2 \sin^2\theta\,, \tag{3.105}$$

(see next section for the full (3+1)-dimensional case).

Despite this simplicity, the implications of the above results are nontrivial. On physical grounds, the isounit $\hat{1} > 0$ permits a direct representation of the nonspherical shapes, as well as all their infinitely possible deformations. By recalling that $O(3)$ is a *theory of rigid bodies*, $\hat{O}(3)$ results to be a *theory of deformable bodies* [63] with fundamentally novel physical applications in the theory of elasticity, nuclear physics, particle physics, crystallography, and other fields [115], [116].

On mathematical grounds, we have equally intriguing novel insights. To see them, one must first understand the background isogeometry $\hat{E}^{III}(r,\hat{\delta},\hat{\Re})$ which unifies all possible conics in $E(r,\delta,\Re)$ [115], as mentioned earlier. To be explicit in this important point, the geometric differences between (oblate or prolate) ellipsoids and (elliptic or hyperbolic) paraboloids have mathematical sense when projected in our Euclidean space $E(r,\delta,\Re)$. However all these surfaces are geometrically unified with the perfect isosphere in $\hat{E}(r,\hat{\delta},\hat{\Re})$.

These geometric occurrences permits the unification of $O(3)$ and $O(2.1)$, as well as of all their infinitely possible isotopes, as reviewed in Sect.s 3.2 and 3.3.

Even greater differentiations between the Lie and Lie-Santilli theories occur

in their representations when both considered in conventional spaces over conventional fields (see Sect. 3.1) because of the change in the eigenvalue equations due to the nonunitarity of the map indicated in Sect. 3.1, from the familiar form $H\psi = E^\circ\psi$, to the isotopic form $H{*}\hat\psi = \hat E{*}\hat\psi \equiv E\hat\psi$, $E^\circ \neq E$), thus implying generalized weights, Cartan tensors and other structures studied earlier.

The first differences emerge in the spectrum of eigenvalues of $\hat o(2)$ and $o(2)$. In fact, the $o(2)$ algebra on a conventional Hilbert space *solely* admits the spectrum $M = 0, 1, 2, 3$ (as a necessary condition of unitarity). For the covering $\hat o(2)$ isoalgebra on an isohilbert space with isotopic element $T = \mathrm{Diag.}\,(g_{11}, g_{22})$, the spectrum is instead given by

$$\hat M = g_{11}{}^{-1/2} g_{22}{}^{-1/2} M, \quad M = 0, 1, 2, \dots, \quad g_{kk} \neq 0\,, \tag{3.106}$$

and, as such, it can acquire *continuous* values in a way fully consistent with the condition, this time, of *isounitarity*. For the general $\hat O(3)$ case see also the detailed studied of ref.s [116].

It should be noted that the isounit for the polar coordinate underlying spectrum (3.106) is precisely the fact of M, $\hat 1_\theta = g_{11}{}^{-1/2} g_{22}{}^{-1/2}$, and this illustrates the remark of Sect. 3.1 to the effect that the isorepresentations coincide with conventional representation when referred to the proper isospace over the appropriate isofield. In fact, the spectrum $M = 0, 1, 2, \dots$ computed with respect to the unit 1 and the isospectrum $\hat M = \hat 1_\theta M$ computed with respect to the isounit $\hat 1_\theta$ are equivalent.

Despite the above *mathematical* equivalence, the *physical* implications are far reaching because, as stressed in Sect. 3.1, the representation which has physical meaning is that in our space, Eq. (3.106). This implies the possibility that the eigenvalues of the angular momentum which have been believed to admit only discrete values $M = 0, 1, 2, \dots$, can also admit *continuous* values $\hat M$ for particles in interior conditions, e.g., a neutron inside a neutron star (see [116] for all physical aspects).

Similarly, the unitary irreducible representations of $su(2)$ are characterize the familiar eigenvalues

$$J_3\,\psi = M\,\psi, \quad J^2\,\psi = J\,(J+1)\,\psi, \quad M = J, J-1, \dots, -J, \quad J = 0, \tfrac{1}{2}, 1, \dots \,. \tag{3.107}$$

Three classes of irreducible isorepresentation of $\hat{su}(2)$ were identified in [76] which, for the adjoint case, are given by the following generalizations of Pauli's matrices: (1) *Regular isopauli matrices*

$$\hat{\sigma}_1 = \Delta^{-\frac{1}{2}}\begin{pmatrix} 0 & g_{11} \\ g_{22} & 0 \end{pmatrix} \quad \hat{\sigma}_2 = \Delta^{-\frac{1}{2}}\begin{pmatrix} 0 & -i\,g_{11} \\ +i\,g_{22} & 0 \end{pmatrix},$$

$$\hat{\sigma}_3 = \Delta^{-\frac{1}{2}}\begin{pmatrix} g_{22} & 0 \\ 0 & -g_{11} \end{pmatrix},$$

$$\hat{T} = \mathrm{diag.}(g_{11}, g_{22}), \quad \Delta = \det \hat{T} = g_{11}\, g_{22} > 0,$$

$$[\hat{\sigma}_i, \hat{\sigma}_j]_{\xi} = i\, 2\Delta^{\frac{1}{2}}\, \epsilon_{ijk}\, \hat{\sigma}_k\, .$$

$$\hat{\sigma}_3 \hat{\times} |\hat{b}> = \pm \Delta^{\frac{1}{2}} |\hat{b}>, \quad \hat{\sigma}^{\hat{2}} \hat{\times} |\hat{b}> = 3\Delta |\hat{b}>,. \tag{3.108}$$

(2) *Irregular isopauli matrices*

$$\hat{\sigma}_1{}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1, \quad \hat{\sigma}_2{}' = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix} = \sigma_2,$$

$$\hat{\sigma}_3{}' = \begin{pmatrix} g_{22} & 0 \\ 0 & -g_{11} \end{pmatrix} = \Delta \hat{1} \sigma_3,$$

$$[\hat{\sigma}'_1 \hat{,} \hat{\sigma}_2{}']_{\xi} = 2\, i\, \hat{\sigma}_3{}',$$

$$[\hat{\sigma}'_2, \hat{\sigma}_3{}']_{\xi} = 2\, i\, \Delta\, \hat{\sigma}_1{}', \qquad [\hat{\sigma}'_3, \hat{\sigma}_1{}']_{\xi} = 2\, i\, \Delta\, \hat{\sigma}'_2,$$

$$\hat{\sigma}'_3 \hat{\times} |\hat{b}> = \pm \Delta |\hat{b}>,$$

$$\hat{\sigma}^{\hat{2}} \hat{\times} |\hat{b}> = \Delta(\Delta + 2) |\hat{b}>. \tag{3.109}$$

(3) *Standard isopauli matrices*

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & \lambda \\ \lambda^{-1} & 0 \end{pmatrix}, \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i\lambda \\ i\lambda^{-1} & 0 \end{pmatrix},$$

$$\hat{\sigma}_3 = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & -\lambda \end{pmatrix},$$

$$T = \mathrm{diag.}(\lambda, \lambda^{-1}), \quad \lambda \neq 0, \qquad \Delta = \det T = 1,$$

$$[\hat{\sigma}''_i, \hat{\sigma}''_j]_{\xi} = i\, \epsilon_{ijk}\, \hat{\sigma}''_k,$$

$$\hat{\sigma}''_3 \hat{\times} | \hat{b} > = \pm | \hat{b} > , \qquad \hat{\sigma}''^{2} \hat{\times} | \hat{b} > = 3 | \hat{b} > . \tag{3.110}$$

The primary differences in the above isorepresentations are the following. For the case of the regular isorepresentations, the isotopic contributions can be factorized with respect to the conventional Lie spectrum. For the irregular case this is no longer possible. Finally, for the standard case we have conventional spectra of eigenvalues under a generalized structure of the matrix representations, as indicated by the appearance of a completely unrestricted, integro-differential function λ.

The regular and irregular representations of $\hat{o}(3)$ and $s\hat{u}(2)$ are applied to the angular momentum and spin of particles under extreme physical conditions, such as an electron in the core of a collapsing star. The standard isorepresentations are applied to conventional particles evidently because of the preservation of conventional quantum numbers [116]. The appearance of the isotopic degrees of freedom then permit novel physical applications, that is, applications beyond the capacity of Lie's theory even for the simpler case of preservation of conventional spectra (see Section 3.G).

A *spectrum-preserving map* from the conventional representations J_g of a Lie-algebra L with metric tensor g to the covering isorepresentations $\mathfrak{J}_{\hat{g}}$ of the Lie-Santilli algebra $\hat{L}$ with isometric $\hat{g} = Tg$ and isounit $\hat{1} = T^{-1}$ is important for physical application. It is called the *Klimyk rule* [27] and it given by

$$\mathfrak{J}_{\hat{g}} = J_g \times \hat{P} , \qquad \hat{} = k \hat{1}, \quad k \in F , \tag{3.111}$$

under which Lie algebras are turned into Lie-Santilli isoalgebras

$$J_i \times J_j - J_j \times J_i = C_{ij}{}^k J_k \equiv (\mathfrak{J}_i \hat{\times} \mathfrak{J}_j - \mathfrak{J}_j \hat{\times} \mathfrak{J}_i) k^{-1} \hat{T} = C_{ij}{}^k k^{-1} T \mathfrak{J}_k , \tag{3.112}$$

that is,

$$\mathfrak{J}_i \hat{\times} \mathfrak{J}_j - \mathfrak{J}_j \hat{\times} \mathfrak{J}_i = C_{ij}{}^k \mathfrak{J}_k , \tag{3.113}$$

thus showing the preservation of the original structure constants.

However, by no means, the Klimyk rule can produce *all* Lie-Santilli isoalgebras, because the latter are generally characterized by *nonunitary* transforms of conventional algebras, with a general variation of the structure constants.

Nevertheless, the Klimyk rule is sufficient for a number of physical applications where the preservation of conventional quantum numbers is

important, because it permits the identification of one specific and explicit form of standard isorepresentations with "hidden" degrees of freedom represented by the isotopic element T available for specific uses.

For instance, the standard isopauli matrices permit the reconstruction of the exact isospin symmetry in nuclear physics under electromagnetic and weak interactions [76], or the construction of the *isoquark theory* [99] with all conventional quantum numbers, yet with an *exact confinement* (i.e., possessing an identically null probability of tunnel effects for free isoquarks because of the incoherence between the interior and exterior Hilbert spaces), and other novel applications.

II.3.7. Lorentz-Santilli isosymmetry and its isodual

We now study the *isotopies* $\hat{L}(3.1)$ *of the Lorentz symmetry* L(3.1), introduced by Santilli in paper [59], then studied in detail in monograph [114] at the classical level, in monograph [116] at the operator level, and today called *Lorentz-Santilli isosymmetry.*

Consider the line element in Minkowski space $x^2 = x^\mu \eta_{\mu\nu} x^\nu$, $\mu, \nu = 1, 2, 3, \hat{4}$, with local coordinates $x = \{ x^1, x^2, x^3, x^4 \}$, $x^4 = c_o t$, and metric η = diag. (1 1, 1, −1). Its simple invariance group, the six-dimensional Lorentz group L(3.1), is characterized by the (ordered sets of) parameters given by the Euler's angles and speed parameter, $w = \{ w_k \} = \{ \theta, v \}$, $k = 1, 2, \ldots, 6$, and generators $X = \{ X_k \} = \{ M_{\mu\nu} \}$, in their known fundamental representation (see, e.g., [31], [32]).

Suppose now that the Minkowskian line element is lifted into the most general possible nonlinear-integral form verifying the conditions of Class III

$$x^{\hat{2}} = x^\mu \hat{g}_{\mu\nu}(x, \dot{x}, \ddot{x}, \ldots) x^\nu, \qquad \det \hat{g} \neq 0, \qquad \hat{g} = \hat{g}^\dagger, \qquad (3.114)$$

which represent: all modifications of the Minkowski metric as encountered, e.g., in particle physics; conventional exterior gravitational line elements with $\hat{g} = \hat{g}(x)$, such as the full Schwarzschild line element; all its possible generalizations for the interior problem; etc.

The explicit form of the simple, six-dimensional invariance of generalized line element $x^{\hat{2}}$ was first constructed by Santilli [59] by following the space-time version of Steps 1 to 5 of the preceding section. Step 1 is the identification of the fundamental isotopic element T via the factorization of the Minkowski metric, $\hat{g} = T\eta$ which, under the assumed conditions, can always be diagonalized into the form

$$\hat{T} = \mathrm{diag.}(g_{11}, g_{22}, g_{33}, g_{44}), \ \hat{T} = \hat{T}^{\dagger}, \ \det T \neq 0. \qquad (3.115)$$

The fundamental isounit of the theory is then given by $\hat{1} = T^{-1}$.

Step 2 is the lifting of the conventional numbers into the isonumbers via the isofields $\hat{\mathfrak{R}}(\hat{n},+,*)$, $\hat{n} = n\,\hat{1}$ (which are different than those of $\hat{O}(3)$ because of the different dimension of the isounit).

Step 3 is the construction of the isospaces in which the isometric $\hat{g}$ is properly defined, which are given by the isominkowski spaces $\hat{M}(x,\hat{g},\hat{\mathfrak{R}})$. The reader should keep in mind that, when $\hat{g}$ is a conventional Riemannian metric, isospaces $\hat{M}(x,\hat{g},\hat{\mathfrak{R}})$ are not the Riemannian spaces $R(x,\hat{g},\mathfrak{R})$ because the basic units of the two spaces are different.

Step 4 is also straightforward. The *Lorentz-Santilli isosymmetry* $\hat{L}(3.1)$ is characterized by the isotransformations

$$\hat{O}(3.1): \qquad x' = \hat{\Lambda}(\hat{w}) \hat{\times} x = \tilde{\Lambda}(w)\, x, \qquad (3.116)$$

verifying the basic properties

$$\hat{\Lambda}^{\dagger}\, \hat{g}\, \hat{\Lambda} = \hat{\Lambda}\, \hat{g}\, \hat{\Lambda}^{\dagger} = \hat{1}\, \hat{g}\, \hat{1}, \text{ or } \tilde{\Lambda}^{\dagger}\, \hat{g}\, \tilde{\Lambda} = \tilde{\Lambda}\, \hat{g}\, \tilde{\Lambda}^{\dagger} = \hat{g},$$

$$\hat{\mathrm{D}}\mathrm{et}\, \hat{\Lambda} = [\,\mathrm{Det}\,(\hat{\Lambda}\, T)\,] = \pm \hat{1}. \qquad (3.117)$$

It is easy to see that $\hat{L}(3.1)$ preserves the original connectivity properties of $L(3.1)$ (see [61] for a detailed study). The connected component $\hat{SO}(3.1)$ of $\hat{L}(3.1)$ is characterized by $\hat{\mathrm{D}}\mathrm{et}\,\hat{\Lambda} = +\hat{1}$ and has the structure [loc. cit.]

$$\hat{\Lambda}(w) = \prod\nolimits^{*}_{k=1,2,\dots,6} e_{\xi}^{\,i\, X_k * \hat{w}_k} = \{ \prod\nolimits_{k=1,2,\dots,6} e^{\,i\, X_k\, T\, w_k} \}\, \hat{1}, \qquad (3.118)$$

where the parameters are the conventional ones, the generators X_k are also the conventional ones in their fundamental representation and the isotopic element T is given by Equations (3.23). The discrete part of $\hat{L}(3.1)$ is characterized by $\hat{\mathrm{D}}\mathrm{et}\,\hat{\Lambda} = -\hat{1}$, and it is given by the *space-time isoinversions* [loc. cit.]

$$\hat{\pi} \hat{\times} x = \pi\, x = -r, x^4), \qquad \hat{\tau} \hat{\times} x = \tau\, x = (r, -x^4). \qquad (3.119)$$

Again, under the assumed conditions for T, the convergence of infinite series (3.58) is ensured by the original convergence, thus permitting the explicit calculation of the symmetry transformations in the needed explicit, finite form. Their space components have been given in the preceding Section 3.E. The additional *Lorentz-Santilli isoboosts* were explicitly computed for the first time in [59], yielding the

expression for all possible isometrics $\hat{g}$

$$x'^1 = x^1, \quad x'^2 = x^2,$$

$$x'^3 = x^3 \cosh[v(g_{33}g_{44})^{\frac{1}{2}}] - x^4 g_{44}(g_{33}g_{44})^{-\frac{1}{2}} \sinh[v(g_{33}g_{44})^{\frac{1}{2}}] =$$

$$= \hat{\gamma}(x^3 - g_{33}^{-1/2} g_{44}^{1/2} \hat{\beta} x^4),$$

$$x'^4 = -x^3 g_{33}(g_{33}g_{44})^{-\frac{1}{2}} \sinh[v(g_{33}g_{44})^{\frac{1}{2}}] + x^4 \cosh[v(g_{33}g_{44})^{\frac{1}{2}}]$$

$$= \hat{\gamma}(x^4 - g_{33}^{1/2} g_{44}^{-1/2} \beta x^3), \tag{3.120}$$

where

$$x^4 = c_0 t, \quad \beta = v / c_0, \quad \hat{\beta} = v^k g_{kk} v^k / c_0 g_{44} c_0,$$

$$\cosh[v(g_{33}g_{44})^{\frac{1}{2}}] = \hat{\gamma} = (1 - \hat{\beta}^2)^{-\frac{1}{2}}, \quad \sinh[v(g_{33}g_{44})^{\frac{1}{2}}] = \hat{\beta}\hat{\gamma}. \tag{3.121}$$

Again, one should note: (A) the unrestricted character of the functional dependence of the isometric $\hat{g}$; (B) the remarkable simplicity of the final results whereby the explicit symmetry transformations are merely given by plotting the values $g_{\mu\mu}$ in Equations (3.120); and (C) the generally nonlinear–nonlocal–noncanonical character of the isosymmetry.

The *isocommutation rules* when the generators $M_{\mu\nu}$ are in their regular representation can also be readily computed and are given by [loc. cit.]

$$\hat{O}(3.1): \; [M_{\mu\nu} \hat{,} M_{\alpha\beta}] = \hat{g}_{\nu\alpha} M_{\beta\mu} - \hat{g}_{\mu\alpha} M_{\beta\nu} - \hat{g}_{\nu\beta} M_{\alpha\mu} + \hat{g}_{\mu\beta} M_{\alpha\nu}, \tag{3.122}$$

with *isocasimirs*

$$C^{(0)} = \hat{1}, \quad C^{(1)} = \tfrac{1}{2} M_{\mu\nu} \times T \times M^{\mu\nu} = M \hat{\times} M - N \hat{\times} N,$$

$$C^{(3)} = \tfrac{1}{2} \epsilon^{\mu\nu\rho\sigma} M_{\mu\nu} \times T \times M_{\rho\sigma} = -M \hat{\times} N,$$

$$M = (M_{12}, M_{23}, M_{31}), \quad N = (M_{01}, M_{02}, M_{03}) \tag{3.123}$$

The classification of all possible isotopes $\hat{SO}(3.1)$ was also done in the original construction [59] via the realizations of the isotopic element

$$\hat{T} = \text{diag.}\,(\pm b_1^2, \pm b_2^2, \pm b_3^2, \pm b_4^2), \quad b_\mu > 0, \tag{3.124}$$

where the b's are the characteristic functions of the interior medium, resulting in:

(1) The conventional orthogonal symmetry SO(4) for T = diag. (1, 1, 1, −1);
(2) The conventional Lorentz symmetry SO(3.1) for T = diag. (1, 1, 1, 1);
(3) The conventional de Sitter symmetry SO(2.2) for T = diag. (1, 1, −1, 1);
(4) The isodual $SO^d(4)$ for T = diag. (−1, −1, −1, 1);
(5) The isodual $O^d(3.1)$ for T = − diag. (1, 1, 1, 1);
(6) The isodual $SO^d(2.2)$ for T = diag. (−1, −1, 1, −1);
(7) The infinite family of isotopes $\hat{SO}(4) \approx SO(4)$ for T = diag. $(b_1^2, b_2^2, b_3^2, -b_4^2)$;
(8) The infinite family of isotopes $\hat{SO}(3.1) \approx SO(3.1)$ for T = diag. $(b_1^2, b_2^2, b_3^2, b_4^2)$;
(9) The infinite family of isotopes $\hat{SO}(2,2) \approx SO(2.2)$ for T = diag. $(-b_1^2, b_2^2, b_3^2, b_4^2)$;
(10) The infinite family of isoduals $\hat{SO}^d(4) \approx SO^d(4)$ for T = diag. $(-b_1^2, -b_2^2, -b_3^2, b_4^2)$;
(11) The infinite family of isoduals $\hat{SO}^d(3.1) \approx SO(3.1)$ for T = − diag. $(b_1^2, b_2^2, b_3^2, b_4^2)$;
(12) The infinite family of isoduals $\hat{SO}^d(2.2) \approx SO^d(2.2)$ for T = diag. $(b_1^2, -b_2^2, -b_3^2, -b_4^2)$.

On the basis of the above results, Santilli submitted the *conjecture* that *all simple Lie algebra of the same dimension over a field of characteristic zero in Cartan classification can be unified into one single abstract isotopic algebra of the same dimension.*

The above conjecture was proved by Santilli for the cases n = 3 and 6. A theorem unifying all possible fields into the isoreals was proved by Kadeisvili et al [26] in the expectation of such general unification. The conjecture has been recently studied by Tsagas [124] for the non-exceptional case.

In the above presentation we have shown that the lifting of the Lorentz symmetry can be naturally formulated for Class III. Nevertheless, whenever dealing with physical applications, the isotopic element is restricted to have the positive- or negative-definite structure $T = \pm\text{diag.}\,(b_1^2, b_2^2, b_3^2, b_4^2)$, thus restricting the isotopies to $\hat{SO}(3.1) \approx SO(3.1)$ and $\hat{SO}^d(3.1) \approx SO^d(3.1)$.

The operator realization of the Lorentz-Santilli isoalgebra is the following. The linear four-momentum admits the isotopic realization [116]

$$p_\mu \hat{\times} |\hat{\psi}\rangle = -i\partial_\mu |\hat{\psi}\rangle = -i T_\mu{}^\nu \partial_\nu |\hat{\psi}\rangle . \tag{3.125}$$

Also, for $x_\mu = \eta_{\mu\nu} x^\nu$ (where η is the conventional Minkowski metric). one can show

that $\hat{\partial}_\mu x_\nu = \hat{\eta}_{\mu\nu}$. The fundamental relativistic isocommutation rules are then given by[loc. cit.]

$$[x_\mu \hat{,} p_\nu] = i\,\hat{\eta}_{\mu\nu}\,, \quad [x_\mu \hat{,} x_\nu] = [p_\mu \hat{,} p_\nu] = 0\,, \tag{3.126}$$

The isocommutation rules are then given by

$$\hat{o}(3.1):\ [\,M_{\mu\nu}\,\hat{,}\,M_{\alpha\beta}\,] = i\,(\hat{\eta}_{\nu\alpha}\,M_{\beta\mu} - \hat{\eta}_{\mu\alpha}\,M_{\beta\nu} - \hat{\eta}_{\nu\beta}\,M_{\alpha\mu} + \hat{\eta}_{\mu\beta}\,M_{\alpha\nu})\,, \tag{3.127}$$

thus confirming the isomorphism $\hat{SO}(3.1) \approx SO(3.1)$ for all positive–definite T.

Again, from the analysis of this section one can conclude that the isotopic and conventional transforms and representations are equivalent when each one is formulated in its own space over its own field. However, the physical space time remains that of the conventional Minkowski space, and the isominkowskian representation has therefore a purely mathematical significance.

The implications of the Lorentz–Santilli isosymmetry then emerge in their full light, because it implies a step–by–step isotopic lifting of the special relativity, called *Santilli's isospecial relativity*, outlined in the next section.

II.3.8. Poincare´– Santilli isosymmetry and its isodual.

We now study the most important topic of this paper, the *isotopies* $\hat{P}(3.1) = \hat{L}(3.1)\hat{\times}\hat{T}(3.1)$ *of the Poincaré symmetry* $P(3.1) = L(3.1)\times T(3.1)$, where $T(3.1)$ represents translations in space–time, which were first introduced by Santilli in memoir [67] of 1988, then worked out in details in papers [79], [95], presented classically in monograph [114], quantum mechanically in monograph [116], and today called the *Poincaré–Santilli isosymmetry*.The isospinorial form $\hat{\mathscr{P}}(3.1) = S\hat{L}(2.\hat{C})\times\hat{T}(3.1)$ was worked out in paper [95] (see also monograph [116]). For brevity, we here limit ourselves to a brief outline of the nonspinorial case.

A generic element of $\hat{P}(3.1)$ can be written $\hat{A} = (\,\hat{\Lambda}, \hat{a}\,)$, $\hat{\Lambda} \in \hat{O}(3.1)$, $\hat{a} \in \hat{T}(3.1)$ with isocomposition

$$\hat{A}' \hat{\times} \hat{A} = (\,\hat{\Lambda}', \hat{a}')\hat{\times}(\,\hat{\Lambda}, \hat{a}\,) = (\,\hat{\Lambda}\hat{\times}\hat{\Lambda}', \hat{a} + \hat{\Lambda}'\hat{\times} a'\,)\,, \tag{3.128}$$

The realization important for physical applications is that via conventional generators in their adjoint representation for a system of n particles of non–null mass m_a

$$X = \{X_k\} = \{\textstyle\sum_a (x_a^\mu p_{a\nu} - x_a^\nu p_{a\mu}),$$

$$P = \{P_\mu\} = \{\textstyle\sum_a p_a\}, k = 1, 2, .10, \tag{3.129}$$

and conventional parameters $w = \{w_k\} = \{v, \theta, a\}$, where v represents the Lorentz parameters, θ represents the Euler's angles, and a characterizes conventional space-time translations.

The connected component of the isopoincaré group is given by

$$\hat{P}(3.1): \quad x' = \hat{A} \hat{\times} x, \qquad \hat{A} = \hat{\prod}_k e_\xi^{\,iX_k w_k} = \{\prod_k e^{iX_k T w_k}\} \times \hat{1}, \tag{3.130}$$

where the isotopic element T and the Lorentz generators $M_{\mu\nu}$ have the same realization as for $\hat{O}(3.1)$. The primary different with isosymmetries $\hat{O}((3.1)$ is the appearance of the isotranslations

$$\hat{T}(3.1) \hat{\times} x = \{e_\xi^{\,iP\,\eta\,a}\} \hat{\times} x = e_\xi^{\,iP\,\hat{g}\,a}\} \hat{\times} x = x + \hat{a}, \quad \hat{T}(3.1) \hat{\times} p \equiv 0. \tag{3.131}$$

The application of the isotheory of Sect.s 3.2, 3.3 and 3.4 then permits the proof of the following:

Theorem 3.8 (General Poincare'-Santilli isogroup [95]): *The "general Poincaré-Santilli isogroup" of Class III as characterized by the isotheory is given by the twentytwo-dimensional isotransforms*

$$x' = \hat{A} \hat{\times} x \qquad \text{Lorentz-Santilli isotransf.}, \tag{3.132a}$$

$$x' = x + a_o B(s, x, \dot{x}, \ddot{x}, ...), \quad \text{isotransl.}, \tag{3.132b}$$

$$x' = \hat{\pi}_r \hat{\times} x = (-r, x^4), \quad \text{space isoinv.}, \tag{3.132c}$$

$$x' = \hat{\pi}_t \hat{\times} x = (r, -x^4), \qquad \text{time isoinv.}, \tag{1.132d}$$

$$\hat{1} \to \hat{1}' = n^2 \hat{1}, \ \hat{\eta} \to \hat{\eta}' = \hat{\eta} / n^2, \ \text{isoselfdilation.} \tag{3.132e}$$

$$\hat{1} \to \hat{1}^d = -\hat{1}, \ \hat{\eta} \to \hat{\eta}^d = -\eta, \quad \text{isoduality} \qquad ($$

$$\hat{X} \to \hat{X}^d = -\hat{X}, \quad \hat{w} \to \hat{w}^d = -\hat{w} \qquad 3.132f)$$

where the *B-functions are given by the expansions*

$$B_\mu = b_\mu + a^\alpha [\, b_\mu \hat{,} P_\alpha \,] / 1! + a^\alpha a^\beta [\, [\, b_\mu \hat{,} P_\alpha \,] \hat{,} P_\beta \,] / 2! + \ldots\ldots \tag{3.133}$$

Isotransforms (3.132a)–3.132d) are a direct consequence of the preceding analysis. The last two isotransforms (3.132e) and (3.132f) originate from the *isoscalar character* of the line element, that is, its structure $(x - y)^{\hat{2}}$ = number×isounit $\in \hat{R}(\hat{n},\hat{+},\hat{\times})$. In fact, the same isotransforms cannot be defined for the conventional line element $(x - y)^2$ = number $\in R(n,+,\times)$.

To understand th dimensionality, we note that the Poincaré–Santilli isosymmetry of Class III is given by the iseselfdual direct product $\hat{P}(3.1)\hat{\times}\hat{P}^d(3.1)$. By recalling that isotopic and isodual structures are independent (because defined on independent spaces over independent fields), this yields double the conventional dimensionaly, that is, *twenty-dimensions*. One additional dimensionality has been discovered by Santilli [101] via his isoscalar transforms (3.132e) which, when combined with the independent isotopic image via isodualities (1.32f), yields *twentyy-two dimensions*.

Note that the isoduality and isodilation of the unit *do not exist* for the conventional transform, and this explains the reason of the transition from a ten- to an eleven-dimensional structure in each isospaces and its isodual.

The classical [114] and operator [116] realizations of the isopoincaré theory are similar, For brevity we review here the latter which is characterized by the *isocommutation rules*

$$[\, M_{\mu\nu} \hat{,} M_{\alpha\beta} \,] = i\,(\, \hat{\eta}_{\nu\alpha} M_{\beta\mu} - \hat{\eta}_{\mu\alpha} M_{\beta\nu} - \hat{\eta}_{\nu\beta} M_{\alpha\mu} + \hat{\eta}_{\mu\beta} M_{\alpha\nu} \,),$$

$$[\, M_{\mu\nu} \hat{,} P_\alpha \,] = i\,(\, \hat{\eta}_{\mu\alpha} P_\nu - \hat{\eta}_{\nu\alpha} P_\mu \,),$$

$$[\, P_\mu \hat{,} P_\nu \,] = 0\,, \mu, \nu, \alpha, \beta = 1, 2, 3, 4, \tag{3.134}$$

with *isocenter* is characterized by the *isocasimirs*

$$C^{(0)} = \hat{1}\,, \quad C^{(1)} = P^{\hat{2}} = P \times \hat{T} \times P = P_\mu\, \hat{g}^{\mu\nu} P_\nu\,,$$

$$C^{(2)} = \hat{W}^{\hat{2}} = \hat{W}_\mu\, \hat{g}^{\mu\nu} \hat{W}_\nu\,, \quad \hat{W}_\mu = \epsilon_{\mu\alpha\beta\rho} J^{\alpha\beta} \hat{\times} P^\rho\,. \tag{3.135}$$

The *restricted isotransformations* occur when the isotopic element $\hat{T}$ is constant.

An important application of the isotranslation is the characterization of the so-called *isoplane-waves* on $\hat{M}(x,\hat{\eta},\hat{\mathfrak{R}})$

$$\hat{\psi}(x) = e_{\hat{\xi}}^{\,ip\hat{\times}x} = \hat{1}\, e^{\,ip\hat{T}x} = \hat{1}\, e^{\,ip_\mu \hat{g}^{\mu\nu} x_\nu} = \hat{1}\, e^{\,i(p_k b_k^2 x_k - p_4 b_4^2 x_4)}, \tag{3.136}$$

which are solutions of the isotopic field equations, represents electromagnetic waves propagating within inhomogeneous and anisotropic media such as out atmosphere and offer quite intriguing predictions for experimentally verifiable <novel> effects, that is, effects beyond the predictive or descriptive capacities of the Poincaré symmetry.

As one can see, the verification of total conservation laws (for a system assumed as isolated from the rest of the universe), is intrinsic in the very structure of the isosymmetry. In fact, the generators are the conventional ones and, since they are invariant under the action of the group they generate, they characterize conventional total conservation laws. The simplicity of reading off the total conservation laws from the generators of the isosymmetry should be compared with the rather complex proof in conventional gravitational theories.

The *isodual Poincaré–Santilli isosymmetry* $\hat{P}^d(3.1)$ is characterized by the isodual generators $X_k{}^d = - X_k$, the isodual parameters $w_k{}^d = - w_k$, and the isodual isotopic element $T^d = -T$, resulting in the change of sign of isotransforms. This implies a novel *law of universal invariant under isoduality* which essentially state that any system which is invariant under a given symmetry is automatically invariant under its isodual. In turn, this law apparently permits novel advances in the study of antiparticles [116].

The significance of the Lie–Santilli isotheory for gravitation is illustrated by the following important property of the isosymmetry $\hat{P}(3.1)$ which evidently follows from of Theorem 3.5:

Theorem 3.9 (Universal Poincare´– Santilli Isosymmetry) [95]: *The general Poincaré–Santilli isotransforms of Theorem 3.6 with Class III isounits* $\hat{1}(x, \dot{x}, \ddot{x}, \dots) = T^{-1}$ *leave invariant all infinitely possible (3+1)-dimensional intervals with isometrics* $\hat{\eta}(x, \dot{x}, \ddot{x}, \dots) = T(x, \dot{x}, \ddot{x}, \dots)\eta$, *where* η *is the Minkowski metric,*

$$(x - y)^{\hat{2}} = [(x - y)^{\mu} \hat{\eta}_{\mu\nu}(x, \dot{x}, \ddot{x}, \dots) (x - y)^{\nu}] \hat{1}\,, \tag{3.137}$$

It should be noted that we have restricted the formulation of the above theorem to Class III on precautionary grounds pending topological studies on broader classes. In fact, a physically more significant formulation of Santilli's isopoincaré symmetry is that of Class IV which include Class III plus the zero of the isounits which represent gravitational singularities [116]. A yet broader definition would be that of Class V with unrestricted, thus, discontinuous isounits.

It is an instructive exercise for the interested reader to verify that the

isodistance (3.75) is indeed left invariant by all isotransforms (3.70).

As it is well known, each invariance of a space-time separation has profound physical implications. In fact, Theorem 3.7 essentially characterizes an isoselfdual covering of the special relativity for interior conditions worked out by Santilli at the classical [114] and operator [116] level, and known as *Santilli's isospecial relativity.*

The latter is a *covering* of the conventional relativity because: 1) it applies for much broader systems (nonlocal-integral and variationally nonselfadjoint systems); 2) it is constructed via structurally more general methods (the isotopic ones), and 3) it contains the conventional relativity as a particular case for $\hat{1} = 1$.

Yet, the two relativities coincide at the abstract level by conception and construction for Class I isotopies [114], [116]. This ultimate identity of the special and isospecial relativities assures the axiomatic consistency of the new relativity because criticisms on the latter ultimately result to be criticisms on Einstein's theories.

To outline some of the main result and implications of the isospecial relativity, the Lorentz-Santilli isosymmetry has numerous applications for *interior conditions*, such as [116]: direct representation of locally varying speeds of light

$$c = c_0\, g_{44}{}^{1/2} = c_o / n\,, \tag{3.138}$$

where n is the ordinary index of refraction; exact-numerical representation of the difference in cosmological redshift between quasars and their associated galaxies, which is reduced to the *decrease of the speed of light* within the quasar's huge chromospheres; and other predictions in various fields.

The isoinversions permit the regaining of exact discrete symmetries when conventionally broken, such as the regaining of the exact space-parity under *weak* interactions by embedding the symmetry breaking terms in the isounit.

The invariance under isoduality (isoselfduality) assures the consistency of the isodual representation of antimatter, evidently because the same invariant holds identically for both matter and antimatter.

Moreover, the invariance under isotopic dilation (isoselfdilation) confirms the direct representation of the locally varying character of the speed of light. For instance, light propagating within homogeneous and isotropic water is represented by the isotopic element $\hat{T}$ with elements $g_{\mu\mu} = n^2$. The isoinvariant then reduces identically to the conventional invariant

$$\begin{aligned}(x - y)^{\hat{2}} &= [\,(x - y)^{\mu}\,(\eta_{\mu\mu} / n^2)\,(x - y)^{\nu}\,] \times (n^2\, 1) \equiv \\ &\equiv (x - y)^{\mu}\, \eta_{\mu\mu}\,(x - y)^{\mu} \times 1 \end{aligned} \tag{3.139}$$

This permits the resolution of the problematic aspects of the special relativity in water, such as the apparent violation of the principle of causality, or the violation of the relativistic sum of speeds (because the sum of two light speeds in water does not yield the speed of light in water for the conventional Lorentz symmetry, but the sum is correct for the Lorentz–Santilli isotransforms).

Rather intriguingly, the quantity n in isoinvariance (3.139e) is non–null but otherwise arbitrary. Santilli's isospecial relativity therefore predicts in a natural way that the speed of light is a *local* quantity which arbitrarily smaller or bigger than the speed of light in vacuum. In fact, $c = c_0/n$ is smaller than c_0 in ordinary media such as water, but it is predicted to be bigger than c_0 in other conditions, such as in the hyperdense media inside hadrons or inside stars. For all these aspects and related references, see [116].

The implications of Theorem 3.9 for gravitation alone are far reaching, and we can only indicate them here without treatment. To begin, the theorem includes as particular cases the conventional Riemannian metric $\hat{\eta}(x, \dot{x}, \ddot{x}, ...) = g(x)$,. *The Poincaré–Santilli isosymmetry therefore provides the universal invariance of all infinitely possible exterior gravitation in vacuum.* More generally, Theorem 3.9 includes all infinitely possible signature–preserving modifications of the Minkowski and Riemannian metrics for *interior* problems.

The simplicity of this universal invariance should also be kept in mind and compared with the known complexity of other approaches to nonlinear symmetries. In fact, one merely *plots* the $g_{\mu\mu}$ elements in isotransforms (3.98), (3.120) and (3.132) without any need to compute anything, because the invariance of general separation (3.75) is ensured by the theorem. For numerous examples, see [95], [116].

Moreover, Theorem 3.9 implies *the unification of the special and general relativities.* [116]. After all, the unification is a necessary prerequisite for the very achievement of the universal symmetry of gravitation. Santilli achieves the unification by factorizing the Minkowski metric in any given exterior Riemannian metric g(x),

$$g(x) = T_{gr}(x)\,\eta\,, \tag{3.140}$$

and then by embedding the *gravitational isotopic element* $T_{gr}(x)$ in the *gravitational isounit*

$$\hat{1}_{gr}(x) = [\,T_{gr}(x)\,]^{-1}\,. \tag{3.141}$$

The Poincaré–santilli isosymmetry with the above isounit then evidently unifies the general and special relativity.

Note *the necessity of the representation of gravitation in the isominkowskian space* $\hat{M}(\hat{x},\hat{\eta},\hat{R})$, $\hat{\eta}(x) = T_{gr}(x)\eta$, $\hat{1}_{gr} = [T_{gr}(x)]^{-1}$, *for the achievement of such a unification.* In fact, no isosymmetry can be formulated in Riemannian spaces, as clear from the review of this section. This implies the formulation of gravity in an *isoflat* space. In fact, the space $\hat{M}(\hat{x},\hat{\eta},\hat{R})$, being an isotopy of the Minkowski space, preserves the geometric properties of the latter, including flatness, yet possesses the same metric as the Riemannian space, thus permitting a novel characterization of gravity.

Another implication of Theorem 3.9 is a *novel quantization of gravity* [116] which is based on *the embedding of gravitation in the unit of relativistic quantum mechanics without any need of a Hamiltonian.* In fact, the quantization is achieved via the lifting of the four-dimensional Minkowskian unit I = diag. (1, 1, 1, 1) of relativistic quantum mechanics into $\hat{1}_{gr}(x)$. As the reader can verify, the operator treatment of the Poincaré-santilli isosymmetry reviewed above *is* a quantum version of gravity for $\hat{1}(x, \dot{x}, \ddot{x}, ...) = \hat{1}_{gr}(x)$. The commutativity of the linear momentum, Eq.s (3.134b) confirms the novel achievement of a *flat* representation of gravity in terms of the *Riemannian* metric which emerges as the *structure functions* $g(x) \equiv \hat{\eta}$ of Eq.s (3.134).

The *isotopic* quantization gravity, called by Santilli *quantum-iso-gravity,* has itself rather deep implications. Recall that quantum gravity is afflicted by serious problems of consistency, such as the lack of invariance of the unit with consequential inapplicability to actual measures, the general lack of preservation of Hermiticity in time with consequential lack of observables, etc. [116]. Quantum-iso-gravity avoids all these problems *ab initio.* In fact, the isotopies assure that quantum-iso-gravity is as axiomatically consistent as relativistic quantum mechanics. After all, the two theories coincide at the abstract level because, from the local Minkowskian character of gravity, $T_{gr}(x)$ is necessarily positive-definite.

Also, *Theorem 3.9 predicts antigravity for elementary antiparticles in the field of matter* [86], [116]. In essence, calculations show that the gravitational force for *antimatter-antimatter systems in vacuum* characterized by $P^d(3.1)$ is *attractive* in the same way as for *matter-matter systems in vacuum* characterized by P(3.1). However, *antimatter-matter systems in vacuum experience a gravitational repulsion,* because they are characterized by the *projection* of $P^d(3.1)$ in the space of P(3.1) (see [116] for details. Note that these results are derived by the simplest possible case of Theorem 3.7, that in vacuum for $\hat{1} = 1$ and $\hat{1}^d = -1$.

Theorem 3.9 has even greater implications in cosmology, because it implies a new conception of the universe called *Santilli's isocosmology* [116], which is based on the universal isosymmetry $U = \hat{P}_I(3.1) \times \hat{P}^d{}_{II}(3.1)$ and implies that, at the limit of equal amounts of matter and antimatter, all total characteristics of the universe are

identically null, including null total energy, null total mass, null total time, etc.

This renders the act of creation of the universe more mathematically treatable then the "big bang" and other models, because the total characteristics remain null before and after the creation in Santilli's isocosmology, while for the "big bang" and other models we have the creating of the immensity of the universe literally from "nothing" with evident large discontinuity at creation.

Recall that the Poincaré symmetry provides the invariance only of relativistic classical and quantum systems. The significance of the Poincaré-Santilli isosymmetry is then illustrated by the fact that it provides the invariance of all (well behaved) infinitely possible, linear or nonlinear, local or nonlocal, Hamiltonian or nonhamiltonian, relativistic or gravitational, exterior or interior, classical or operator, and local or cosmological systems.

II.3.9. Mathematical and physical applications.

Lie's theory is known to be at the foundation of virtually all branches of mathematics. The existence of intriguing and novel applications in mathematics originating from the Lie-Santilli theory is then beyond scientific doubts.

With the understanding that mathematical studies are at their first infancy, the isotopies have already identified new branches of mathematics besides isoalgebras, isogroups and isorepresentations. We here mention: the new branch of number theory dealing with isonumbers; the new branch of functional isoanalysis dealing with isospecial isofunctions, isotransforms and isodistribution; the new branch of topology dealing with the integro-differential topology; the new branch of the theory of manifold dealing with isomanifolds and their intriguing properties; and so on. It is hoped that interested mathematicians will contributed to these novel mathematical advances which have been identified and developed until now mainly by physicists, except a few exceptions.

Lie's theory in its traditional linear-local-canonical formulation is also known to be at the foundation of all branches of contemporary physics. Profound physical implications due to the covering, nonlinear-nonlocal-noncanonical Lie-Santilli theory cannot therefore be dismissed in a credible way.

With the understanding that these latter applications too are at the beginning and so much remains to be done, we have recalled after Theorem 3.7 some of the implications of the isotheory. We refer the interested reader to monographs [116], [118] for several additional applications and experimental verifications in nuclear physics, particle physics, astrophysics, cosmology, superconductivity, theoretical biology and other fields.

PART III:

SANTILLI'S ISO–GRAND–UNIFICATION AND ISO–COSMOLOGY

1997

III.1. INTRODUCTION

While this second edition was about to be released for print in early 1997, Santilli achieved in Ref.s [1,2,3] of Part III the apparently first, axiomatically consistent inclusion of gravitation in unified gauge theories, resulting in a novel *Iso-Grand-Unification.*

This result is important for this volume because it constitute the climax of the entire chain of studies of the preceding Parts I and II which can therefore be justified only at the level of grand unification. The new Iso-Grand-Unification has therefore implications for all the studies considered in this volume, all the way to a deeper understanding of the isoselfdual cosmology of Ref. [116] of Part II.

In particular, the new grand unification provide the ultimate confirmation for the need of the main lines of studies considered in the preceding parts, such as: the isominkowskian representation of gravity without curvature; the isodual representation of antimatter; the Lie-Santilli isotheory in general, and the Poincaré-Santilli isosymmetry in particular; the synthesis of current relativities into one single unified formulation; the isospecial relativity (Ref. [1] of Part III and [116] of Part II); and other advances.

In this Part III we shall outline these advances by following *verbatim* Ref. [3] for the grand unification and Ref. [116] of Part II, plus the updates of Ref.s [1, 2] of Part III for cosmological profiles. An advanced knowledge on isotopies is needed for a technical understanding of this Part III.

In essence, Santilli studies in Ref. [3] the structural incompatibilities for an axiomatically consistent inclusion of gravitation in the unified gauge theories of electroweak interactions due to (for brevity, see [loc. cit.] for all historical references):

1) Curvature. Electroweak theories are essentially structured on *Minkowskian* axioms, while gravitational theories are currently formulated via *Riemannian* axioms, a disparity which is magnified at the operator level because of known technical difficulties of quantum gravity, e.g., to provide a PCT theorem

comparable to that of electroweak interactions.

2) Antimatter. Electroweak theories are *bona fide* relativistic theories, thus characterizing antimatter via *negative-energy* solutions, while gravitation characterizes antimatter via *positive-definite* energy-momentum tensors.

3) Fundamental space-time symmetries. Electroweak interactions are based on the axioms of the special relativity, thus verifying the fundamental *Poincaré symmetry*, while such a basic symmetry is absent in contemporary gravitation.

Without any claim of being unique, Ref. [3] presents, apparently for the first time, a conceivable resolution of the above structural incompatibilities via the use of the following new methods:

A) Isotopies. A baffling aspect in the inclusion of gravity in unified gauge theories is their apparent geometric incompatibility despite their individual beauty and experimental verifications.

The view here considered is that the above structural incompatibility is not necessarily due to insufficiencies in Einstein's field equations, but rather to *insufficiencies in their mathematical treatment*.

Stated in plain language, Santilli claims that the achievement of axiomatic compatibility between gravitation and electroweak interactions requires a basically new mathematics, that is, basically new numbers, new spaces, new geometries, etc.

The mathematics used for the resolution of the incompatibility due to curvature is the *isomathematics* studied in the preceding Part I at the elementary level and Part II at a more advanced level. The main idea is that presented by Santilli at *VII Marcel Grossmann Meeting on General Relativity* (see Ref. [105] of Part II), and consisting in:

I) the factorization of any given Riemannian metric g = g(x) into the the Minkowski metric η = diag. (1, 1, 1, 1)

$$g(x) = \hat{T}(x) \times \eta\,, \tag{1.1}$$

where $\hat{T}(x)$ is a 4×4 matrix which is positive-definite (from the locally Minkowskian character of Riemann);

II) the reconstruction of the Riemannian geometry with respect to the isounit

$$\hat{1}(x) = [\hat{T}(x)]^{-1}, \qquad (1.2)$$

iii) the lifting of all associative products among generic quantities A, B (i.e., numbers, vector fields, operators, etc.) into the familiar isoassociative form

$$A \hat{\times} B = A \times \hat{T} \times B, \qquad (1.3)$$

under which $\hat{1} = \hat{T}^{-1}$ is the correct left and right new unit.

This yields *Santilli's isominkowskian gravity*, namely, gravitation represented with the *isominkowskian geometry* (see Ref.s [1,2] of Part III) which admits all possible Riemannian metric and conventional field equations, but refers them to the generalized unit $\hat{1}(x)$.

Note that curvature is contained in the isotopic term $\hat{T}(x)$, because η is flat. Therefore, *the reformulation of Riemannian metrics* $g(x) = \hat{T}(x) \times \eta$ *with respect to the isounit* $\hat{1} = \hat{T}^{-1}$, *eliminates curvature in isospace, thus rendering gravitation geometrically compatible to the electroweak interactions.*

Ref. [3] also points out that the Iso-Grand-Unification can be derived from the isoselfscalar invariance studied in Sect. II.3.8

$$I \rightarrow n^2 \times I = \hat{1}, \quad \eta \rightarrow \eta' = n^{-2} \times \eta = \hat{\eta}, \qquad (1.4)$$

under which we have the novel invariance of the conventional Minkowskian line element

$$x^2 = (x^\mu \times \eta_{\mu\nu} \times x^\nu) \times I \equiv [x^\mu \times (n^{-2} \times \eta_{\mu\nu}) \times x^\nu] \times (n^2 \times I) \equiv$$

$$\equiv (x^\mu \times \hat{\eta}_{\mu\nu} \times x^\nu) \times \hat{1}, \qquad (1.5)$$

with related invariance of the conventional Hilbert product

$$<\phi | \times | \psi > \times I \equiv <\phi | \times (n^{-2}) \times | \psi > \times (n^2 \times I) \equiv <\phi | \times \hat{T} \times | \psi > \times \hat{1}. \quad (1.6)$$

The isominkowskian formulation of gravity then follows when the parameter n is enlarged to be function of the coordinates, as it occurs in the transition from Abelian to non-Abelian gauge theories.

In short, the classical and operator isominkowskian representation of gravity is rooted in novel, fundamental symmetries of the Minkowski and Hilbert spaces, respectively.

The reader should be aware that the proposed resolution of incompatibility i) works best where it is needed most, at the operator level. In fact, the operator

formulation of the isominkowskian representation of gravity the that of Sect. II.2.13 when the isounit acquires the gravitational value (1.2). As such, it verifies all abstract axioms and physical laws of conventional relativistic quantum mechanics (see Ref. [105], Part II). The emerging new theory, called *operator isogravity,* merely consists in embedding gravity in the unit of relativistic quantum mechanics.

The reader should be aware that the above results imply *the abandonment of curvature for the characterization of gravitation in favor of broader notions.*

Rather than being surprising or a peculiarity of grand unifications, the need to abandon curvature for isoflat treatments is dictated by a number of conceptual, theoretical and experimental evidence.

On conceptual grounds, the assumption in the celebrated "bending of light" that light follows the curved geodesic of the local gravitational field is in contradiction with the experimental evidence that gravitation attracts all forms of energy, as it is the case for a mass in free vertical fall. In fact, light does carry energy, thus resulting in the prediction of bending within a gravitational field which is *double* the experimental value, one due to the curvature and the other due to ordinary gravitational attraction.

This leaves no other scientific alternative than that of *either* assuming an actual curvature of space-time *without* gravitational attraction, *or* assuming gravitational attraction *without* curvature of space-time. Santilli elected the second alternative because the former is disproved by masses in free fall.

On theoretical grounds, the following fundamental theorem was proved by Santilli in memoir [101] of Part II:

Theorem 3.1: *The basic units of space and time are not invariant for all possible geometries with non-null curvature.*

Proof. Recall from Sect. II.3.8 that the line element of the isominkowski space is "directly universal", that is, including as particular cases in the fixed coordinates of the observer the metrics of all possible curved geometries, thus including Riemannian geometry. Recall also from Sect. II.3.8 that the universal symmetry of said line element, the Poincaré-Santilli isosymmetry, is *noncanonical* at the classical level and *nonunitary* at the operator level. Theorem 3.1 then follows because noncanonical and nonunitary transforms are well known not to preserve the basic units of space and time by their very definition. **q.e.d.**

The above theorem confirms the need for the isominkowskian representation of gravitation without conventional curvature in a way completely independent from the requirements of a grand unification. In fact, rules (1.1)-(1.3) permit the formulation of gravity under the uncompromisable condition of

possessing invariant units of space and time, although of generalized character, as it is the case for the special relativity. Note that this is the very fundamental element of the covering isospecial relativity [1].

On experimental grounds, the need to abandon curvature for the characterization of gravity following the above advances is beyond scientific doubts. Recall that the isopoincaré symmetry *is* the time evolution of geometries with arbitrary curvature. It then follows that *geometries with conventional non-null curvatures cannot be applied to real measurements in a scientifically consistent way*, because one of the fundamental conditions for measurements is precisely the invariance of the basic units.

As an example, it is not possible to conduct the measurement, say, of a length with a stationary meter changing in time. The attempt at preserving old knowledge via the assumption that the entire system changes in time is flawed, e.g., for measurements of length related to far away objects which, as such, are outside the influence of the local gravitational field.

In summary, to our best knowledge, no other known theory can resolve the incompatibility between electroweak interactions and gravitation due to curvature as well as the shortcomings of Theorem 3.1, thus confirming the need for Santilli's isominkowskian representation of gravity without curvature.

B) Isodualities. Structural incompatibility 2) is only the symptom of deeper problems in the contemporary treatment of antimatter outlined in the introduction of Sect. II.2.

The view submitted in Ref. [3] of Part III is that, as it is the case for curvature, the resolution of the above general shortcomings requires another, basically novel mathematics.

That proposed by Santilli is the *isodual isomathematics* of Part II based on the isodual map of the isominkowskian representation of gravity

$$\hat{1} > 0 \rightarrow \hat{1}^d = -\hat{1}^\dagger = -\hat{1} < 0, \tag{1.7a}$$

$$g(x) = \hat{T}(x) \times \eta \rightarrow g^d(x) = \hat{T}^d \times^d \eta^d = -g(x), \tag{1.7b}$$

$$A \hat{\times} B = A \times \hat{T} \times B \rightarrow A \hat{\times}^d B = A \times \hat{T}^d \times B = -A \hat{\times} B. \tag{1.7c}$$

The latter approach implies *Santilli's isodual isominkowskian representation of gravity for antimatter* (Ref.s [1,2] of Part III) which is based on a *negative-definite energy-momentum tensor*, thus characterizing antiparticles with negative-energy as it is the case for electroweak interactions.

The electroweak interactions themselves are also re-interpreted via the

isodual theory in the following way. The conventional *retarded* solutions are solely used for the representation of *particles*, while the *advanced* solution are solely used for the representation of *antiparticles*. Since advanced solution are usually discarded nowadays, the above isodual reformulation of electroweak interactions recovers all conventional numerical results.

In conclusion, in Santilli's Iso-Grand-Unification, antiparticles are treated in both gravitation and electroweak interactions via the isodual theory, thus resolving incompatibility 2).

The latter theory is also based on new symmetries, the *isodual symmetry*

$$I \rightarrow -n^2 \times I = \hat{1}^d, \quad \eta \rightarrow \eta' = n^{-2} \times \eta = \hat{\eta}, \tag{1.8}$$

under which we have the additional novel invariance of the conventional Minkowskian line element

$$x^2 = (x^\mu \times \eta_{\mu\nu} \times x^\nu) \times I \equiv [x^\mu \times (-n^{-2} \times \eta_{\mu\nu}) \times x^\nu] \times (-n^2 \times I) \equiv$$
$$\equiv (x^\mu \times \hat{\eta}^d_{\mu\nu} \times x^\nu) \times \hat{1}^d, \tag{1.9}$$

with corresponding novel symmetry of the conventional Hilbert product

$$< \phi | \times | \psi > \times I \equiv < \phi | \times (-n^{-2}) \times | \psi > \times (-n^2 \times I) \equiv$$
$$\equiv < \phi | \times \hat{T}^d \times | \psi > \times \hat{1}^d, \tag{1.10}$$

This establishes the isodual theory of antimatter on solid foundations at both classical and operator levels (see Part II for details on isodual theories).

To our best knowledge, no other approach can resolve the incompatibility due to antimatter between electroweak and gravitational interactions, thus confirming Santilli's isodual representation of antimatter.

C) Isotopies of the Poincare' symmetry and their isoduals. The most severe problems of compatibility between gravitation and electroweak interactions for both matter and antimatter exist precisely were expected, in the fundamental space-time symmetries, because of the disparity indicated earlier caused by the validity of the Poincaré symmetry for electroweak interactions and its absence for gravitation.

The latter problem is resolved by the *Poincaré-Santilli isosymmetry* $\hat{P}(3.1)$ of Sect. II.3.8 constructed for the gravitational isounit $\hat{1}(x) = \eta \times g^{-1}$, and its isodual $\hat{P}^d(3.1)$ for the isodual gravitational isounit $\hat{1}^d(x^d)$. In fact, the electroweak

interactions of the Iso–Grand–Unification for matter are formulated on an isominkowski space $\hat{M}(\hat{x},\hat{\eta},\hat{R})$ with gravitational unit $\hat{1}(x)$, thus requiring the transition from the Poincaré to the Poincaré–Santilli isosymmetry which then becomes "universal", that is, applicable to both. A similar structure emerges for antimatter under isodualities.

In conclusion, to our best knowledge, Santilli's isotopies and isodualities of the Poincaré symmetry are the only known approach capable of resolving the structural incompatibility between gravitation and electroweak interactions due to the fundamental space–time symmetries.

When adding the isodual isopoincaré symmetry for antimatter, the total space–time isosymmetry of the Iso–Grand–Unification is given by the isoselfdual product

$$\hat{S}_{Tot}{}^{Isoselfd} = \hat{\mathcal{P}}(3.1) \times \hat{\mathcal{P}}^{d}(3.1) =$$

$$= [\, \hat{SL}(2, \hat{C}) \hat{\times} \hat{T}(3.1) \,] \times [\, \hat{SL}^{d}(2, \hat{C}^{d}) \hat{\times}^{d} \hat{T}^{d}(3.1) \,] \equiv \hat{S}^{d}{}_{Tot}{}^{Isoself}, \tag{1.11}$$

where $\hat{\mathcal{P}}(3.1)$ characterizes the unified theory for matter and $\hat{\mathcal{P}}^{d}(3.1)$ that for antimatter. Everything follows in a unique and unambiguous way.

To understand the implications of this Part III, the reader should know that Santilli discovered the above universal isosymmetry in the study of the symmetry of the *conventional* Dirac equation

$$(\, \gamma^{\mu} \times p_{\mu} + i \times m \,) \times | \psi > = 0, \tag{1.12a}$$

$$\gamma^{k} = \begin{pmatrix} 0 & \sigma_{k} \\ -\sigma_{k} & 0 \end{pmatrix}, \quad \gamma^{4} = i \times \begin{pmatrix} I_{s} & 0 \\ 0 & -I_{s} \end{pmatrix}, \tag{1.12b}$$

$$\backslash \psi > = \text{Column}\, \{ | \psi_{1} >, | \psi_{2} > \}. \tag{1.12c}$$

In essence, Santilli noted that *the negative unit is present in the very structure of Dirac's gamma matrices.* Dirac was forced to invent the "hole theory" because *negative-energy* solution referred to *positive* units behave in an unphysical way. Santilli argued that *negative-energy* solutions when referred to *negative* units behave in a fully physical way. He therefore constructed his isodual mathematics precisely around Dirac's negative unit $-I_{s} = I^{d}{}_{s}$.

This permitted the following isodual reformulation of the conventional equation (see ref.s [3] of Pat III and [106] of Part II)

$$(\, \gamma^{\mu} \times p_{\mu} + i \times m \,) \times | \psi > = 0, \tag{1.13a}$$

$$\gamma^k = \begin{pmatrix} 0 & \sigma_k \\ \sigma^d{}_k & 0 \end{pmatrix}, \quad \gamma^4 = i \times \begin{pmatrix} I_s & 0 \\ 0 & I^d{}_s \end{pmatrix}, \tag{1.13b}$$

$$|\Psi> = \text{Column}\ \{|\psi>, |\psi>^d\}, \tag{1.13c}$$

which is fully symmetric for particle and antiparticles and, does not require second quantization for antiparticles, an occurrence which is expected for a theory of antiparticles which begins at the classical *Newtonian* level (see Part II).

The following additional property discovered by Santilli [loc. cit.] has fundamental relevance all the way to cosmology:

Lemma 1.1: *The conventional Dirac gamma matrices are isoselfdual, i.e., invariant under isoduality,*

$$\gamma^\mu \rightarrow \gamma'^\mu = -\gamma^{\dagger\mu} \equiv \gamma^\mu. \tag{1.14}$$

The above property established that *the belief held throughout this century that the Poincaré symmetry is the global symmetry of Dirac's equation is inconsistent.* In fact, the Poincaré symmetry *is not* isoselfdual, $P(3.1) \neq P^d(3.1)$. As such, it cannot possibly be the symmetry of isoselfdual laws such as Eq.s (1.2) or (1.13).

In this way Santilli constructed the following true isoselfdual symmetry of the conventional Dirac equations,

$$S_{tot}{}^{Dirac6} = \mathcal{P}(3.1) \times \mathcal{P}^d(3.1) \equiv S^d{}_{tot}{}^{Dirac}, \tag{1.15}$$

which is *twenty-dimensional* because isodual spaces and parameters are independent from conventional; structures.

In particular, *Santilli's studies disprove the additional belief held throughout this century that the spin in Dirac's equation is characterized by a 4×4-dimensional representation.* In fact, the correct spin symmetry of Dirac'xs equation is $SU(2) \times SU^d(2)$. As a result, the spin for particles does indeed remain characterized by the *two-dimensional* Pauli's matrices σ_k and that of antiparticles by their isodual Pauli matrices $\sigma^d{}_k = -\sigma_k$.

Next, Santilli discovered in memoir [101] of Part II the additional *isoselfscalar invariance of Dirac's equation,* i.e., its invariance under transforms (1.4)-(1.6) which, when combined with their isoduals (1.8)-(1.10), brings the total

dimensionality of the symmetry of Dirac's equation to *twenty-two independent parameters.*

The final step was the lifting of symmetry (1.15) to isounits with nontrivial functional dependence which lead the fundamental isoselfdual isosymmetry (1.11).

The reader is suggested to meditate a moment on the implications of the above discoveries. First, they imply that contemporary mathematics as currently formulated *is not* applicable to antimatter.

Second, the above results establish that the conception throughout this century of gravitation as being represented by curvatures is fundamentally flawed and should be replaced with broader notions.

Third, the same results establish that the classical and quantum physics of this century has been constructed with *incomplete space-time symmetries*, evidently because they are not isoselfdual.

Finally, the reader should not be surprised that the new invariances (1.4)–(1.6) and (1.8)–(1.10) have remained undetected throughout this century. In fact, their identification required the prior discovery of *new numbers*, first the isonumbers with arbitrary positive units for invariances (1.4)–(1.6), and then the additionally new isodual isonumbers with arbitrary negative units for invariances (1.8)–(1.10).

III.2. ISO–GRAND–UNIFICATION

In a communication to the *VIII Marcel Grossmann meeting on General Relativity* scheduled to be held in Jerusalem next June, 1997, Santilli (Ref. [3] of Part III) presented a basically novel Iso-Grand-Unification with the axiomatically consistent inclusion of gravity.

In particular, Ref. [3]: presents the formal derivation of the new unification; provides very simple means for its explicit construction for each given gravitational metric; proves the invariance of the theory under nonunitary transforms; identifies the new grand unification as a realization of the theory of "hidden variables"; and indicates that it is along the historical teaching by Einstein, Podolsky and Rosen on the lack of completion of quantum mechanics.

The isotopies of gauge theories were first studied in the 1980's by Gasperini [4a] followed by Nishioka [4b], Karajannis and Jannussis [4c] and others, and ignored thereafter. These studies were defined on conventional spaces over conventional fields and via the conventional differential calculus. As such, they resulted not to be invariant following the recent studies of memoirs [100,101] of Part II.

In Ref. [3] Santilli introduced, apparently for the first time, the *isotopies of gauge theories*, or *isogauge theories* for short, formulated in an invariant way,

that is, on isospaces over isofields and characterized by the isodifferential calculus (see Sect. II.2.7). The *isodual isogauge theories* were apparently introduced in Ref. [3] for the first time.

The essential mathematical methods needed for an axiomatically consistent and invariant formulation of the *isogauge theories for matter* are the following ones, heron assumed as known from Part II, all having the same basic isounit $\hat{1}(x, ...)$:

1) Isofields of isoreal numbers $\hat{R}(\hat{n},+,\hat{\times})$ or isocomplex numbers $\hat{C}(\hat{c},+,\hat{\times})$ and related novel isonumber theory.

2) Isominkowski spaces $\hat{M}(x\hat{c},\hat{\eta},\hat{R})$ equipped with Kadeisvili isocontinuity and Tsagas-Sourlas isotopology of Sect. II.2.5 (see also the recent studies by Aslander and Keles, Ref. [5a] of Part III). A more technical formulation of the isogauge theory can be done via the isobundle formalism on isogeometries recently reached by Vacaru (Ref. [5b] of Part III) which is under study at this writing.

3) Isodifferential calculus of Sect. II.2.7

4) Isofunctional isoanalysis of Sect. II.2.6

5) Isominkowskian geometry which is a symbiotic unification of the Minkowskian and Riemannian geometries available in its latest formulation in Ref. [2] of Part III.

6) Hadronic mechanics outlined in Sect. II.2.13 (see memoir [101] of Part II for its latest presentation).

7) The Lie-Santilli isotheory studied in details in Part II.

Once the reader has acquires a technical knowledge of the above methods, the formulation of the isogauge theories is trivial, and merely consists in putting a "hat" on all symbols and all operation of conventional theories.

The isogauge theory for matter is characterized by a non-Abelian isogauge symmetry $\hat{G}$ which is the isotopic image of the conventional symmetry G, i.e., the symmetry G reconstructed for general isounits $\hat{1}(x, ...)$. Since the latter are positive-definite, $\hat{G} \approx G$ by conception and construction.

The isosymmetry $\hat{G}$ is characterized by: universal enveloping isoassociative algebras $\hat{\xi}(g)$ where the preservation of the symbol g of conventional algebras indicates the preservation of the original generators and only their rewriting in isospace over isofields); Lie-Santilli algebra $\hat{g}$; and the Lie-Santilli isogroup $\hat{G}$ realized in terms of isounitary operators on a isohilbert space $\hat{\mathcal{H}}$ over the isofield $\hat{C}$;

the isorepresentation theory, etc., *all possessing the same isounit* $\hat{1}(x, ...)$.

The isostates then transform according to the isounitary law

$$\hat{\psi}'(\hat{x}) = \hat{U} \times \hat{\psi}(\hat{x}) = \{ \hat{e}^{-i \times X_k \times \theta(x)_k} \} \hat{\times} \hat{\psi}(\hat{x}) = \{ e^{-i \times X_k \times \hat{T} \times \theta(x)_k} \} \times \hat{\psi}(\hat{x}), \quad (2.1)$$

where $\hat{e}$ is the isoexponentiation and $\hat{x}$ is in the isominkowski space. The nontriviality of the isotheory is then shown in this first step by the appearance of a nonlinear operator $\hat{T}$ in the *exponent* of the isogroup structure.

The isocovariant isoderivative is then defined by

$$\hat{D}_\mu \hat{\psi} = (\hat{\partial}_\mu - \hat{i} \hat{\times} \hat{g} \hat{\times} \hat{A}_\mu{}^k(\hat{x}) \hat{\times} \hat{X}_k) \hat{\times} \hat{\psi}(\hat{x}), \quad (2.2)$$

where one should recall that $\hat{i} = i \times \hat{1}$, $\hat{g} = g \times \hat{1}$, g is the usual electroweak structure constant, $\hat{i} \times \hat{g} \hat{\times} \hat{A}_\mu{}^k \hat{\times} \hat{X}_k = (i \times g \times A_\mu{}^k \times X_k) \times \hat{1}$ and the A's are the potentials. It is easy to see that the above isoderivatives verify the Lie–Santilli axioms in isospace.

It is then easy to prove the isocovariance law

$$(\hat{D}_\mu \hat{\psi})' = \hat{U} \hat{\times} \hat{D}_\mu \hat{\psi}. \quad (2.3)$$

under the usual redefinition, although in isotopic form

$$\hat{A}'_\mu = - \hat{g}^{-1} \hat{\times} (D_\mu \hat{U}) \hat{\times} \hat{U}^{-1}, \quad (2.4)$$

The *Yang–Mills–Santilli isofields* are then defined by

$$\hat{F}_{\mu\nu} = \hat{i} \hat{\times} \hat{g} \hat{\times} [\hat{D}_\mu \hat{,} \hat{D}_\nu] \hat{\psi}(\hat{x}), \quad (2.5)$$

where $[A \hat{,} B] = A \hat{\times} B - B \hat{\times} A = A \times \hat{T} \times B - B \times \hat{T} \times A$ is the Lie–Santilli isoproduct, and they verify properties identical to the conventional ones, only written in isospace.

Finally, we mention that the isogauge theory is derivable from the isoaction in isospace

$$\hat{A} = \int \hat{d}\hat{x} (- \hat{F}_{\mu\nu} \hat{\times} \hat{F}^{\mu\nu}), \quad (2.6)$$

and then the use of the Lagrange–Santilli isoequations (Sect. II.2.10).

Santilli [loc. cit.] insists that the above isogauge theory *is not* a new theory but only a *new realization* of the abstract axioms of the conventional non-Abelian gauge theories. In fact, the isogauge theory coincides at the abstract level with the conventional theory to such an extent that they can be presented with the *same*

conventional symbols, those without "hats", and then subject them to different realizations, the conventional realization for the conventional theory and the isotopic realization for the covering theory.

The *isodual isogauge theories for antimatter* are the anti-isomorphic images of the preceding theories under the isoduality of the totality of the preceding quantities and of their operation.

The total gauge symmetry is therefore $\hat{G} \times G^d$ and it is isoselfdual.

Santilli Iso-Grand-Unification is given by the preceding isogauge theory and its isodual in which the isounit acquires the gravitational value (1.2). The total isosymmetry is therefore the isoselfdual structure

$$\hat{S}_{Tot}{}^{I-G-U} = \{ \hat{\mathcal{P}}(3.1) \hat{\times} \hat{G} \} \hat{\times} \{ \hat{\mathcal{P}}^d(3.1) \hat{\times}^d G^d \} \equiv \hat{S}^d{}_{Tot}{}^{I-G-U} . \qquad (2.7)$$

It is evident that the above theory resolves the axiomatic incompatibilities between electroweak and gravitational interactions identified in Sect. III.1.

Santilli then provided in Ref. [3] a very simple method for the explicit construction of the Iso-Grand-Unification for any given gravitational theory. It is given by the selection of a transformation U which is *nonunitary* on conventional Hilbert spaces, $U \times U^\dagger \neq I$, which is assumed to characterize the isounit according to the rules

$$\hat{1} = U \times U^\dagger = \hat{T}(x) = \eta \times (g(x))^{-1} , \qquad (2.8)$$

yielding specific models, such as that for the Schwarzschild metric

$$\hat{1} = U \times U^\dagger = \mathrm{Diag.}\{ (1 - 2\times M/r), (1 - 2\times M/r), (1 - 2\times M/r), (1 - 2\times M/r)^{-1} \} . \qquad (2.9)$$

The entire isogauge theory then results from the systematic application of the above nonunitary transform, to the *totality* of quantities and operations of the conventional unified theories.

This yields: the isounit $I \rightarrow I' = U \times I \times U^\dagger = U \times U^\dagger = \hat{1}$; the isonumbers $n \rightarrow n' = U \times n \times U^\dagger = n \times \hat{1} = \hat{n}$; the correct isoproduct with the correct Hermiticity and value of the isotopic element, $A \times B \rightarrow U \times (A \times B) \times U^\dagger = A' \times \hat{T} \times B' = A' \hat{\times} B'$, $\hat{T} = (U \times U^\dagger)^{-1} = \hat{1}^{-1} = \hat{T}^\dagger{}_{a}{}' = U \times A \times U^\dagger$, $B' = U \times B \times U^\dagger$; the Lie-Santilli isoalgebras and isogroups; the Yang-Mills-Santilli isofields; and all other aspects of the isogauge theory, now in a specific realization.

Note that the lack of implementation of the above nonunitary lifting to only *one* of the conventional; profiles, e.g., numbers or differential calculus, implies numerous inconsistencies.

Santilli then showed how the resulting Iso-Grand-Unification is indeed

invariant. In fact, any additional nonunitary transform $W \times W^{\dagger} = \hat{1} \neq 1$, can always be factorized into the form $W = \hat{W} \times \hat{T}^{1/2}$ under which it becomes an isounitary transform, $W \times W^{\dagger} = \hat{W} \hat{\times} \hat{W}^{\dagger} = \hat{W}^{\dagger} \hat{\times} \hat{W} = \hat{1}$ yielding the invariance of the Iso-Grand-Unification: $\hat{1} \rightarrow \hat{1}' = \hat{W} \hat{\times} \hat{1} \hat{\times} \hat{W}^{\dagger} \equiv \hat{1}$, $\hat{A} \hat{\times} \hat{B} \rightarrow \hat{U} \hat{\times} (\hat{A} \hat{\times} \hat{B}) \hat{\times} \hat{W}^{\dagger} = \hat{A}' \hat{\times} \hat{B}$; etc.

Note that the invariance implies, not only the preservation of the isotopic structure, but actually the *preservation of the numerical value of the isounit and isotopic element*. Note also that the selection of a nonunitary transform $W \times W^{\dagger} = \hat{1} \neq \hat{1}$ implies the selection of a different gravitational theory.

The above Iso-Grand-Unification results to be a *concrete and specific realization of the theory of "hidden variables"* (see, e.g., Ref. [6a] of Part III). In fact, gravitation is "hidden" in the conventional theory, not only because the conventional and isotopic unified theories coincide at the abstract level, but also because the gravitational isounit preserves all axiomatic properties of the conventional unit

$$\hat{1}^{\hat{n}} = \hat{1} \hat{\times} \hat{1} \hat{\times} \ldots \hat{\times} \hat{1} \equiv \hat{1}, \tag{2.10a}$$

$$\hat{1}^{\hat{\frac{1}{2}}} \equiv \hat{1}, \quad \hat{1} / \hat{1} \equiv \hat{1}, \text{etc.} \tag{2.10b}$$

$$d\hat{1} / dt = \hat{1} \hat{\times} \hat{H} - \hat{H} \hat{\times} \hat{1} = \hat{H} - \hat{H} \equiv 0, \tag{2.10c}$$

while the isoeigenvalues and isoexpectation values of the isounit recover the conventional unit,

$$\hat{1} \hat{\times} | \hat{\psi} > = \hat{T}^{-1} \times \hat{T} \times | \hat{\psi} > = 1 \times | \hat{\psi} L>, \tag{2.11a}$$

$$< \hat{1} > = < \hat{\psi} | \times \hat{T} \times \hat{1} \times \hat{T} \times | \hat{\psi} > / < \hat{\psi} | \times \hat{T} \times | \hat{\psi} > = 1. \tag{2.11b}$$

It then follows that the proposed IGU constitutes an explicit and concrete realization of the theory of "hidden variables" $\lambda = \hat{T}(x, \ldots)$,

$$\hat{H} \hat{\times} | \hat{\psi} > = \hat{H} \times \lambda(x, \ldots) \times | \hat{\psi} > = \hat{E}_{\lambda} \hat{\times} | \hat{\psi} \times = E_{\lambda} \times | \hat{\psi} >, \tag{2.12}$$

when the theory is correctly reconstructed with respect to the new unit $\hat{1} = \lambda^{-1}$ for axiomatic consistency. In particular, von Neumann's Theorem [6b] and Bell's inequalities [6c] do not apply to the above isotopic realization of "hidden variables", evidently because of the nonunitary character of the theory (see Ref. [116] of Part III) for details).

Most intriguingly, Santilli [loc. cit.] shows that his Iso-Grand-Unification is a realization of the teaching by Einstein, Podolsky and Rosen on the "lack of

completion" of quantum mechanics, only applied to the isotopic completion of gauge theories.

We should indicate that the above iso-Grand-Unification is far from having sole mathematical meaning because, in addition to the first, axiomatically consistent inclusion of gravitation in unified gauge theories of electroweak interactions, the isotheory has numerous applications and experimental verifications on existing data and predicts fundamental, basically novel events.

First, the Iso-Grand-Unification is the culmination of all studies by Santilli. As such, all existing applications and experimental verifications in particle physics, nuclear physics, astrophysics and other theories are applications and verifications of the Iso-Grand-Unification (for their outline see memoir [101] of Part II).

moreover, the Iso-Grand-Unification has the following novel predictions:

1) Prediction that antimatter emits a new photon, called *isodual photon* (see Ref. [106] of Part II), which coincides with conventional photon for all interactions except gravitation and which, if confirmed, may allow one day to ascertain whether far away galaxies and quasars are made up of matter or of antimatter.

2) Prediction that all stable isodual particles, that is, the isodual photon, the isodual electron (positron) and the isodual proton (antiproton), experience *antigravity* in the field of matter (defined as the reversal of the sign of the curvature tensor (see also Ref. [106] of Part II). As indicated in Sect. II.2.1, this prediction avoids the usual arguments against antigravity because, e.g., bound states of stable particles and their antiparticles such as the positronium, are predicted to experience attraction in both fields of matter and antimatter, and for other reasons.

3) Prediction that part of the cosmological redshift is of isotopic type, that is, due to the *decrease* of the speed of electromagnetic waves within astrophysical chromospheres (see Ref. [116] of Part II). In particular, Santilli's isominkowskian geometry explains the visible different in the tendency toward the red visible by the naked eye at sunset and sunrise.

4) Prediction that the neutron can be stimulated to decay via subnuclear mechanisms (see Ref. [101], of Part II), with vast implications in nuclear and other technologies;

5) Prediction of a new, clean, subnuclear energy called "hadronic energy" based on stimulated beta decays [loc. cit.]; and other predictions.

Note that the isotopies leave unrestricted the functional dependence of the isounit $\hat{1}(x, \ldots)$, provided that it is positive-definite. Ref. [3] uses only the x-dependence to represent *exterior* gravitational problems in vacuum. The isogauge theory also admits an arbitrary nonlinearity in the velocities and other variables which can be used for the study of *interior* gravitational problems. Also, the

isogauge theory naturally admits a dependence of the isounit on the *wavefunctions* and their derivatives while preserving isolinearity in isospace, as studied in Sect. II.2.13 (thus preserving the superposition principle, as needed for a consistent representation of composite systems). Moreover, the isotheory is a particular case of the broader geno- and hyper-theories (see the concluding comments). Thus, the Iso-Grand-Unification was presented in Ref. [3] as a *particular case* of yet broader formulations.

A scientific appraisal of the new Iso-Grand-Unification requires the knowledge that *the isotopies not only preserve the original axioms, but also the original numerical values* (see Ref. [116] of Part II). As a result, the Iso-Grand-Unification outlined above, not only is impeccable on axiomatic grounds, but also is unquestionable on experimental grounds because it reproduces in isospace all experimental results of both electroweak and gravitational interactions.

In closing, the most significant possibility suggested by Santilli [3] is that *gravitation may have always been present in unified gauge theories. It did creep in un-noticed because embedded where nobody looked for, in the unit of gauge theories.*

III.3. ISOCOSMOLOGY

Santilli's isocosmology is the new cosmology characterized by the total symmetry (see ref. [116], Sect. 9.6 of Part II and the updates in Refs. [1,2,3] of Part III)

$$\hat{S}_{Tot}{}^{Universe} = \{ \hat{\mathcal{P}}(3.1) \hat{\times} \hat{G} \} \times [\hat{\mathcal{P}}^d(3.1) \hat{\times}^d \hat{G}^d] \equiv \hat{S}^d{}_{Tot}{}^{Universe}, \qquad (3.1)$$

where $\hat{\mathcal{P}}(3.1)$ and $\hat{\mathcal{P}}^d(3.1)$ are the space-time symmetries for matter and antimatter, respectively, while $\hat{G}$ and $\hat{G}^d$ characterize direct products of internal symmetries and their isoduals, including gauge, unitary and other symmetries.

It is evident that the above cosmology is based on the use of the isominkowskian (and not the isoriemannian) geometry [2] and the applicable physical laws are those of the isospecial (and not isogeneral) relativity. In this way, Santilli's isopoincaré symmetry, isominkowskian geometry and isospecial relativity become "universal", that is, applicable everywhere throughout the universe.

It should be indicated from the outset that *Santilli's cosmology is the first and only one characterized by a symmetry.* This is evidently due to the fact that gravitation does not possess a symmetry in other treatments, thus preventing the study of a cosmology based on a universal symmetry.

The assumption of basic symmetry (3.1) characterizes the new cosmology

uniquely and unambiguously. Some of the basic properties of the new cosmology are the following:

1) Lack of curvature, as a necessary condition to define a total symmetry for the universe. In fact, the admission of curvature *prohibits* even the formulation of a symmetry for gravitation, as studied in details in Part II. The universe is predicted to be everywhere isoflat. This resolves old controversies, such as the prediction of double bending of light, one for curvature and one for attraction discussed in Sect. III.1.

2) Isoselfduality, i.e., invariance under the map of all quantities and their operations into their anti–Hermitean images. This is mathematically a fundamental property of Santilli's new cosmology with far reaching implications, some of which are indicated below.

3) Equal amounts of matter and antimatter in the universe, evidently as a consequence of the isoselfduality. Needless to say, this is a limit conditions which does not preclude other possibilities, such as total amount of antimatter being smaller than that of matter, in which case however isoselfduality must be broken.

4) Null total characteristic of the universe, i.e., null total energy, null total linear and angular momenta, null total time, Etc.. This is a consequence of characteristics 1) and 2), under the assumption that total quantities are referred to one single observer wether made up of matter or of antimatter.

5) Local notions of time and space. The local character of time is a consequence of the basic units of isosymmetry (3.1) for which the flow of time depends on the value of the fourth component $\hat{1}_{44}$. The latter depends on the local gravitational field as in Eq. (2.9), thus resulting in the indicated local time. Note that questions such as "the age of the universe" have no meaning for Santilli's isocosmology because the answer would be "the age of the universe is identically null". Alternative questions such as "the age of the matter component of the universe" also have no meaning because of the *local* character of time. A question which may have meaning for the isocosmology is "the *average* age of the *matter* component of the universe". However, the latter question too is soon voided by broader versions of the new cosmology, such as the multivalued *hypercosmology* indicated below. The local character of time evidently implies a corresponding local notion of space. In turn, these imply the *mathematical* prediction of a new form of locomotion called "isolocomotion" (see Ref. [115] of Part II) in which motion occurs without any application of a Newtonian force, and via the alteration instead of the local units by means of sufficiently large local energies.

6) Arbitrary local maximal speeds (for matter and of their isoduals for antimatter). In isominkowskian space the maximal causal speed remains that of

light in vacuum c_o [Ref.s [115,116] of Part II and Ref.s [1,2] of Part III). However, the projection of the above speed in our space-time yields the local value $c = c_o/n_4$ where n_4 is unrestricted (except for being positive-definite) and can therefore be smaller, equal or bigger than 1. This fundamental prediction is amply confirmed by the fact that, on real scientific grounds, the speed of light in vacuum is no longer a barrier. In fact, photons traveling within certain guides at speeds bigger than c_o have been measured and then independently confirmed; large masses have been measured to be expelled during astrophysical explosions at speed bigger than c_o and confirmed by additional measurements; according to all phenomenological and experimental data available, the speed of photons inside hadrons, nuclei and stars is bigger than c_o (see Ref. [101] of Part II for details and references); etc..

7) Average speed of light in the universe bigger than that in vacuum (in the matter component and of the isodual light in the antimatter component). We are here referring to the average value of the speed of light throughout the universe, thus including the value in vacuum c_o, *plus* the value $c = c_o/n_4$ within physical media of low density (such as atmospheres and chromospheres in which $c < c_o$) or of high density (such as the media inside hadrons, stars and quasars for which $c > c_o$). Calculations have indicated that the latter dominate over the former resulting in an average speed $c° = \text{Aver}(c) > c_o$.

8) Lack of the missing mass. The universe has been conjectured to have a "missing mass" based on the assumption that the speed of light has everywhere the value c_o in vacuum, resulting in the energy equivalence $E = mc_o^2$. The assumption of an average value of the speed of light $c° > c_o$ evidently implies a revision of the above belief. In fact, the new total energy of the matter component of the universe is given by

$$E_{Tot}^{Univ} = m_{Tot} \times c°^2 = m_{Tot} \times c_o^2 / \text{Aver}(n_4) > E°_{Tot} = m_{Tot} \times c_o^2. \quad (3.2)$$

The above occurrence not only eliminates the need of the missing mass, but actually permits a first estimate of the average speed of light in the universe precisely from the value of the missing mass, according to the expression (eq. (9.6.6), p. 462, Ref. [116] of Part II),

$$n°_4 = \text{Aver}.\, n_4 = [\ (m_{Tot} / (m_{Tot} + m_{Miss})\ ]^{1/2} < 1\,, \quad (3.3)$$

for which $c = c_o/n°_4 > c_o$, as predicted by Santilli.

9) Lack of discontinuity at creation. According to current theories, such as the "big bang", the creation of the universe is based on a large discontinuity, the creation of an immense amount of energy from nothing. In Santilli's isocosmology such a discontinuity is eliminated, because the total characteristics of the universe

were identically null prior to creation and remain identically null after creation. They have been merely separated into equal and opposite values at Creation. This view permits, apparently for the first time, quantitative mathematical-geometrical studies of the Creation thanks precisely to the lack of its discontinuity..

For other characteristics of the novel isocosmology, we refer the interested reader to Ref. [116] of Part II and Ref.s [1,2,3] of Part III.

As final comments, recall that the isomathematics is a particular case of the broader *genomathematics* (see memoir [100] of Part II) which occurs for non-Hermitean generalized units and is used for an axiomatization of irreversibility. In turn, the genomathematics is a particular case of the *hypermathematics* [loc. cit.] which occurs when the generalized units are given by ordered *sets* of non-Hermitean quantities and is used for the representation of multivalued complex systems (e.g.,biological) in irreversible conditions. Evidently both the genomathematics and hypermathematics admit anti-isomorphic images under isoduality.

To understand the dimension of the scientific construction studied in this volume, as well as the amount of novel research it is generating on aspects yet to be investigated, one must therefore keep in mind that the Iso-Grand-Unification and Isocosmology were submitted as *particular cases* in the following chain of lifting of contemporary models:

1) Isodualities of conventional gauge theories and cosmologies for the treatment of antimatter without gravitation in vacuum.

2, 3) Isogauge theories, Isocosmology and their isoduals, for the inclusion of gravity for matter and antimatter in reversible and closed-isolated conditions in vacuum.

4, 5) Genogauge theories, genocosmology and their isoduals, for matter and antimatter, respectively, in irreversible and open interior conditions.

6, 7) Hypergauge theories, hypercosmology and their isoduals, for multivalued, irreversible and open generalizations, e.g., the study of cosmologies inclusive of biological structures.

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ТРУДЫ ИНСТИТУТА ФИЗИКИ

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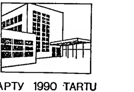
Quasigroups and nonassociative algebras in physics

КВАЗИГРУППЫ И НЕАССОЦИАТИВНЫЕ АЛГЕБРЫ В ФИЗИКЕ



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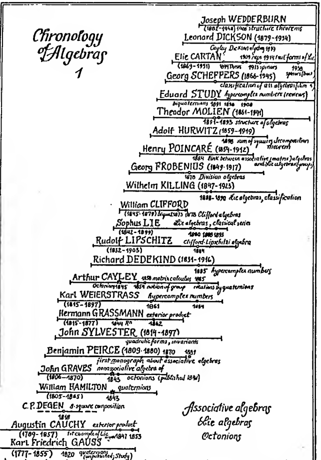
Chronology of Algebras 1
Joseph WEDDERBURN
Leonard DICKSON (1879-1954)
Elie CARTAN
Georg SCHEFFERS (1866-1945)
Eduard STUDY
Theodor MOLIEN (1861-1941)
Adolf HURWITZ (1859-1919)
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Richard DEDEKIND (1831-1916)
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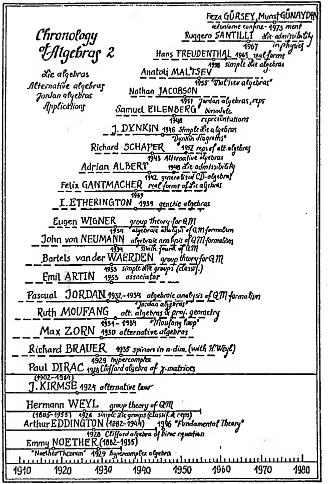
John SYLVESTER (1814-1897)
Benjamin PEIRCE (1809-1880)
John GRAVES
William HAMILTON
C.P. DEGEN
Augustin CAUCHY
Karl Friedrich GAUSS
(1777-1855)
Associative algebras
Lie algebras
Octonions
1790 1800 1810 1820 1830 1840 1850 1860 1870 1880 1890 1900 1910 1920 1930 1940


Chronology of Algebras 2
Lie algebras
Alternative algebras
Jordan algebras
Applications
Feza GÜRSEY, Murat GÜNAYDIN
octonionic confine- 1973 ment
Ruggero SANTILLI Lie-admissibility in physics
1967
Hans FREUDENTHAL 1961 real forms
1958 simple Lie algebras
Anatoli MALTSEV
1955 "Mal'tsev algebras"
Nathan JACOBSON
1951 Jordan algebras, reps
Samuel EILENBERG bimodule representations
1948
J. DYNKIN 1946 Simple Lie algebras
"Dynkin diagrams"
Richard SCHAFER 1952 reps of alt. algebras
1943 Alternative algebras
Adrian ALBERT 1948 Lie admissibility
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Felix GANTMACHER real forms of Lie algebras
1939
I. ETHERINGTON 1939 genetic algebras
Eugen WIGNER group theory for QM
1934 algebraic analysis of QM formalism
John von NEUMANN algebraic analysis of QM formalism
1934 Math. found. of QM
Bartels van der WAERDEN group theory for QM
1933 simple Lie groups (classif.)
Emil ARTIN 1933 associator
Pascual JORDAN 1932-1934 algebraic analysis of QM formalism
"Jordan algebras"
Ruth MOUFANG alt. algebras & proj. geometry
1931-1934 "Moufang loop"
Max ZORN 1930 alternative algebras
Richard BRAUER 1935 spinors in n-dim. (with H. Weyl)
1929 hypercomplex
Paul DIRAC 1928 Clifford algebra of γ-matrices
(1902-1984)
J. KIRMSE 1924 alternative law
Hermann WEYL group theory of QM
(1885-1955) 1926 simple Lie groups (classif. & reps)
Arthur EDDINGTON (1882-1944) 1946 "Fundamental Theory"
1928 Clifford algebra of Dirac equation
Emmy NOETHER (1882-1935)
"Noether Theorem" 1929 hypercomplex algebra
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ABOUT THE BOOK

After reviewing Santilli's isotopies of fields, spaces and algebras, the book presents a detailed description of the isotopies of the symplectic, affine and Riemannian geometries which are nonlinear in the coordinates as well as velocities and accelerations, nonlocal-integral in all variables and nonpotential. The monograph presents Santilli's isogeneral, isospecial and isogalilean relativities for the description of interior dynamical problems (such as a satellite during re-entry) while preserving the axioms of the exterior problem in vacuum. This second edition reviews substantial advances occurred in isotopic relativities during the last three years.

ABOUT THE AUTHOR

J. Vladimir Kadeisvili was born in Georgia and educated in the Russian Mathematics Schools. He is the author of numerous articles in combinatorial topology, functional analysis, Lie's theory over fields of characteristic $p \neq 0$, and other topics. He is primarily known for the classification of the isotopies into five topologically different classes today known as *Kadeisvili's classes.* Prof. Kadeisvili is a member of the International School of Physics of Alma-Ata, Kazakhstan, and a permanent Associate Member of the Division of Mathematics of the Institute for Basic Research, Florida, USA

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